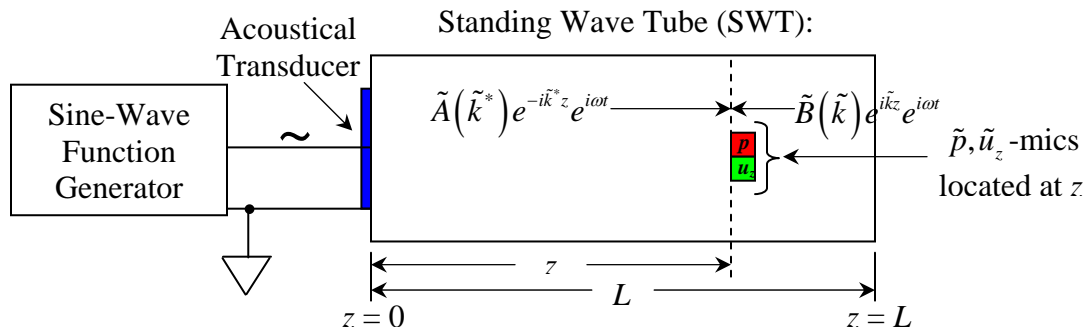


The Acoustical Physics of a Standing Wave Tube

A typical cylindrical-shaped standing wave tube (SWT) {aka impedance tube} of length L and diameter $D \ll L$ with infinitely rigid walls and closed ends is shown in the figure below:



Sound energy is input to the SWT at the position $z = 0$, e.g. using a sine-wave function generator connected to some kind of acoustical transducer, such as a wafer-thin piezo-electric transducer (or a loudspeaker). Ideally-speaking, the transducer should have no frequency-dependent phase-shift(s) relative to the driving sine-wave function generator. However, in the real world, such devices do not exist. At frequencies below the lowest cutoff frequency of the SWT ($f_c^{1,1} \approx 1.84v/\pi D \sim 3300 \text{ Hz}$ for $v = 345 \text{ m/s}$ and $D = 6 \text{ cm}$) only 1-D type plane waves can propagate in the SWT.

Pressure (p) and differential/particle velocity (u_z) microphones are co-located at the “generic” position z along the symmetry axis of the SWT. They are used to record the complex instantaneous total pressure and the instantaneous complex 1-D longitudinal/ z -component of the total particle velocity at that location associated with the presence of right- and left-moving acoustic traveling plane waves propagating in the SWT. The resultant instantaneous complex pressure standing wave at the point z is thus a linear superposition of these two traveling plane waves:

$$\tilde{p}(z, t) = \tilde{A}(\tilde{k}) e^{i(\omega t - \tilde{k}^* z)} + \tilde{B}(\tilde{k}) e^{i(\omega t + \tilde{k} z)} = \tilde{A}(\tilde{k}) e^{-i\tilde{k}^* z} e^{i\omega t} + \tilde{B}(\tilde{k}) e^{i\tilde{k} z} e^{i\omega t} = \left[\tilde{A}(\tilde{k}) e^{-i\tilde{k}^* z} + \tilde{B}(\tilde{k}) e^{i\tilde{k} z} \right] e^{+i\omega t}$$

where the complex, frequency-dependent wavenumber $\tilde{k}(\omega) \equiv k(\omega) + i\kappa(\omega) = \omega/v(\omega) + i\kappa(\omega)$; the $*$ denotes complex conjugation, i.e. $\tilde{k}^*(\omega) \equiv k(\omega) - i\kappa(\omega)$ and $i \equiv \sqrt{-1}$ and thus $-i \equiv -\sqrt{-1}$.

The use of \tilde{k}^* ensures that we are always appropriately mathematically describing decaying exponential attenuation phenomena, i.e. for $z > 0$, using $e^{-\kappa z}$ for both right- and left-traveling waves (as opposed to unphysical, exponentially growing phenomena, i.e. $e^{+\kappa z}$ with distance).

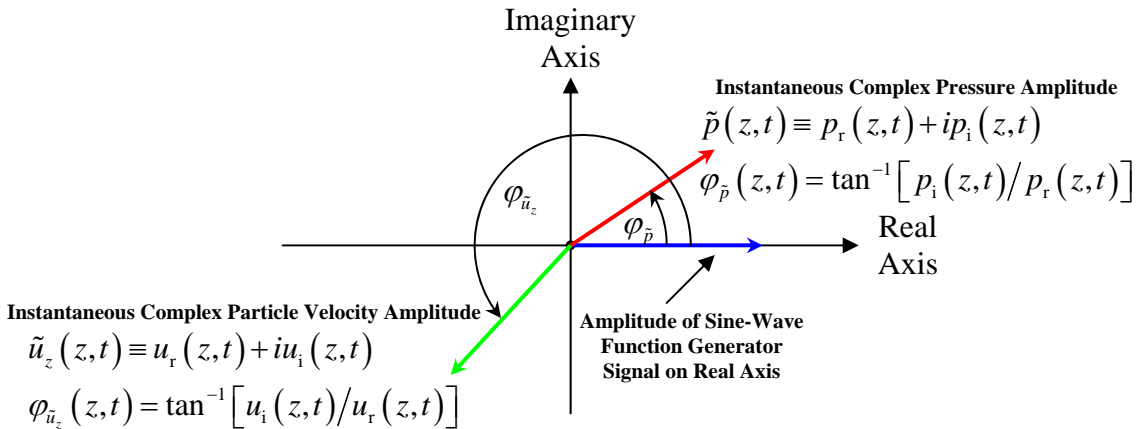
An {extremely} important micro-detail here is that in order to be able to correctly compare theoretical prediction(s) to experimental data, the choice of using $e^{i\omega t}$ vs. $e^{-i\omega t}$ in the theory is in fact not arbitrary. In the UIUC Physics 199POM/498POM SWT experiment, in order to obtain the necessary phase-sensitive information on the complex nature of pressure (p) and 1-D particle velocity (u_z) as a function of frequency, the electrical signals output from the pressure and particle velocity microphone preamplifiers are each input to separate lock-in amplifiers (SRS model # DSP-830) which also use the signal output from the sine-wave function generator as the reference signal for each of the lock-in amplifiers. In our SWT experiment we have explicitly selected 0^o referencing

of the two lock-in amplifiers to the function generator's sine-wave signal, and thus because of the way a lock-in amplifier works, also implicitly means that we have selected the $e^{+i\omega t}$ sign convention. Had we instead selected e.g. 180° referencing of the lock-in amplifier to the sine-wave reference signal, then we would have implicitly instead selected the $e^{-i\omega t}$ sign convention. Because of our $e^{+i\omega t}$ choice in referencing of the lock-in amplifiers to the function generator's sine-wave signal, both the instantaneous complex pressure and instantaneous complex particle velocity process (i.e. rotate) counter-clockwise (CCW) in the complex plane as time increases, since $e^{+i\omega t} = \cos \omega t + i \sin \omega t$ whereas $e^{-i\omega t} = \cos \omega t - i \sin \omega t$.

Generically, the instantaneous complex pressure (SI units: Pascals) and instantaneous complex particle velocity (SI units: m/s) can be written as:

$$\boxed{\tilde{p}(z,t) \equiv p_r(z,t) + ip_i(z,t) = |\tilde{p}(z)| e^{i\phi_p} e^{+i\omega t}} \quad \text{and} \quad \boxed{\tilde{u}_z(z,t) \equiv u_r(z,t) + iu_i(z,t) = |\tilde{u}_z(z)| e^{i\phi_u} e^{+i\omega t}}$$

The real parts of the complex instantaneous pressure $\tilde{p}(z,t)$ and/or complex instantaneous 1-D particle velocity $\tilde{u}_z(z,t)$ are in-phase (if $+ve$) or 180° out-of phase (if $-ve$) relative to the reference signal output from the sine-wave function generator; the imaginary parts of the complex instantaneous pressure $\tilde{p}(z,t)$ and/or complex instantaneous 1-D particle velocity $\tilde{u}_z(z,t)$ are $+90^\circ$ out-of-phase (if $+ve$) or -90° out-of phase (if $-ve$) relative to the reference signal output from the sine-wave function generator, as shown below {for a general case/generic situation} in the so-called phase diagram – i.e. the complex plane, at time $t = 0$:



For small amplitudes, the instantaneous complex pressure and instantaneous 3-D complex vector particle velocity are related to each other via Euler's equation for compressible, inviscid fluid flow {inviscid fluid flow means that any/all viscous/dissipative forces \ll inertial forces}:

$$\boxed{-\rho_o \partial \vec{u}(\vec{r},t) / \partial t = \vec{\nabla} \tilde{p}(\vec{r},t)} \quad \text{where } \rho_o = \text{mass volume density of the fluid and } \boxed{\vec{u}(\vec{r},t) \cdot \vec{\nabla} \{ \vec{u}(\vec{r},t) \} = 0}$$

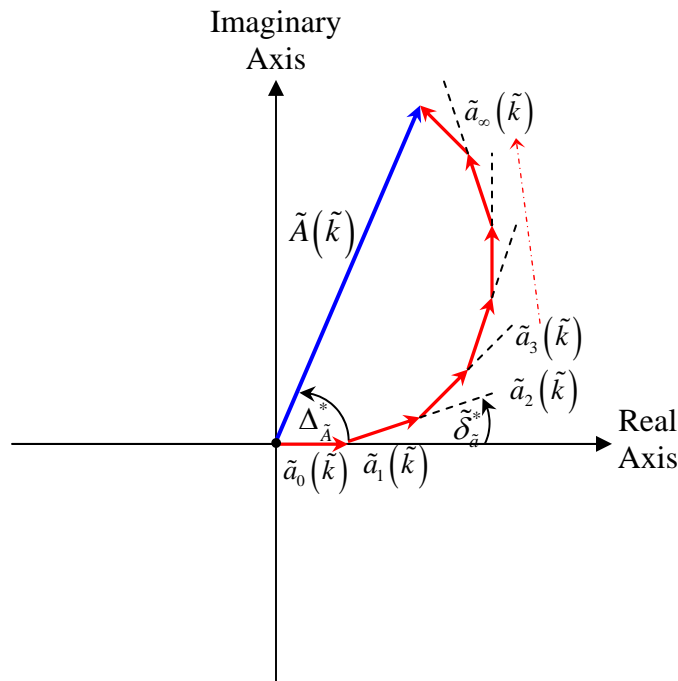
is assumed. For {bone-dry} air at NTP, $\rho_o = 1.204 \text{ kg/m}^3$. For the SWT with 1-D longitudinal particle velocity measurement, the corresponding 1-D Euler's equation for plane waves propagating in the SWT reduces to $\boxed{-\rho_o \partial \tilde{u}_z(z,t) / \partial t = \partial \tilde{p}(z,t) / \partial z}$.

The total instantaneous complex pressure at the position z associated with the presence of a standing **plane** wave in the SWT is the instantaneous linear superposition of an (overall) right-propagating complex traveling plane wave and an (overall) left-propagating traveling plane wave:

$$\tilde{p}(z,t) = \left[\tilde{A}(\tilde{k})e^{-i\tilde{k}z} + \tilde{B}(\tilde{k})e^{+i\tilde{k}z} \right] e^{+i\omega t}$$

The overall instantaneous complex pressure amplitude is in fact a linear superposition of an infinite number of individual right- and left-moving complex traveling pressure waves with complex amplitudes $\tilde{a}_n(\tilde{k})$ and $\tilde{b}_n(\tilde{k})$, $n = 0, 1, 2, 3, \dots, \infty$ respectively, each of which are associated with the sine-wave signal output from the acoustical transducer (located at $z = 0$) at times earlier than $t = 0$. Thus, mathematically the complex amplitudes associated with right- and left-moving complex amplitudes can each be represented by the infinite series $\tilde{A}(\tilde{k}) = \sum_{n=0}^{\infty} \tilde{a}_n(\tilde{k})$ and $\tilde{B}(\tilde{k}) = \sum_{n=0}^{\infty} \tilde{b}_n(\tilde{k})$.

Both of these series representations can be represented graphically via phasor diagrams in the complex plane, e.g. for $\tilde{A}(\tilde{k}) = \sum_{n=0}^{\infty} \tilde{a}_n(\tilde{k})$ as shown in the figure below for $t = 0$:



Precisely on a resonance of the SWT, the individual complex amplitudes $\tilde{a}_n(\tilde{k})$ and $\tilde{b}_n(\tilde{k})$ associated with the individual right- and left-moving traveling waves respectively, are perfectly in phase with each other, i.e. all of the individual relative phases $\delta_{\tilde{a}_n}^* = 2n\pi = 0$, $\delta_{\tilde{b}_n}^* = 2n\pi = 0$ and thus the overall phases $\Delta_{\tilde{A}}^* = \sum_{n=0}^{\infty} \delta_{\tilde{a}_n}^* = 0$ and $\Delta_{\tilde{B}}^* = \sum_{n=0}^{\infty} \delta_{\tilde{b}_n}^* = 0$, graphically corresponding to “straight-line” phasor diagrams for $\tilde{A}(\tilde{k}) = \sum_{n=0}^{\infty} \tilde{a}_n(\tilde{k})$ and $\tilde{B}(\tilde{k}) = \sum_{n=0}^{\infty} \tilde{b}_n(\tilde{k})$.

Explicitly writing out the complex pressure amplitudes $\tilde{A}(\tilde{k}) = \sum_{n=0}^{\infty} \tilde{a}_n(\tilde{k})$ and $\tilde{B}(\tilde{k}) = \sum_{n=0}^{\infty} \tilde{b}_n(\tilde{k})$

associated with the overall right- and left-moving complex plane waves:

$$\tilde{A}(\tilde{k}) = A_o e^{-i\tilde{\delta}_0^*} + A_o e^{-i\tilde{\delta}_1^*} + A_o e^{-i\tilde{\delta}_2^*} + A_o e^{-i\tilde{\delta}_3^*} + \dots = A_o \left[e^{-i\tilde{\delta}_0^*} + e^{-i\tilde{\delta}_1^*} + e^{-i\tilde{\delta}_2^*} + e^{-i\tilde{\delta}_3^*} \dots \right] = A_o \sum_{n=0}^{\infty} e^{-i\tilde{\delta}_n^*}$$

And:

$$\tilde{B}(\tilde{k}) = A_o e^{-i\tilde{\delta}_0^*} + A_o e^{-i\tilde{\delta}_1^*} + A_o e^{-i\tilde{\delta}_2^*} + A_o e^{-i\tilde{\delta}_3^*} + \dots = A_o \left[e^{-i\tilde{\delta}_0^*} + e^{-i\tilde{\delta}_1^*} + e^{-i\tilde{\delta}_2^*} + e^{-i\tilde{\delta}_3^*} \dots \right] = A_o \sum_{n=0}^{\infty} e^{-i\tilde{\delta}_n^*} = \tilde{A}(\tilde{k})$$

The multiplicative phase factor $e^{-i\tilde{\delta}_n^*}$ associated with the n^{th} term in each of the two infinite series arises from the fact that each such contributing wave had to originate at an earlier time, $t_n < 0$ in order for all such waves to arrive simultaneously at the $z = z$ position at the time $t = t$. Note that since $e^{i\omega t}$ rotates complex quantities CCW in the complex plane as the time t increases, the sign in the argument of the $e^{-i\tilde{\delta}_n^*}$ phase factor associated with waves arriving at the $z = z$ position at the time $t = t$ from the earlier time $t_n < 0$ must be negative. Since the elapsed time for n round trips of right- or left-moving waves propagating in the SWT is $\Delta t_n \equiv t - t_n = n(2L)/v = 2nL/v$, then the complex phase shift associated with n round trips of right- or left-moving waves propagating in the SWT is

$$\tilde{\delta}_n^* = \omega \Delta t_n = 2nL\omega/\tilde{v} = 2n\tilde{k}^* L \quad \text{and thus} \quad e^{-i\tilde{\delta}_n^*} = e^{-2in\tilde{k}^* L} = e^{-2in(k-i\kappa)L} = e^{-2\kappa L} e^{-2in\kappa L}$$

Thus:
$$\tilde{A}(\tilde{k}) = A_o + A_o e^{-2\kappa L} e^{-2i\kappa L} + A_o e^{-4\kappa L} e^{-4i\kappa L} + A_o e^{-6\kappa L} e^{-6i\kappa L} + \dots = A_o \sum_{n=0}^{\infty} e^{-2n\kappa L} e^{-2in\kappa L} = \tilde{B}(\tilde{k})$$

Note that since the end walls of the SWT (located at $z = 0$ and $z = L$ respectively) are assumed to be infinitely rigid, we have tacitly/implicitly assumed that no additional phase shift(s) of the right-/left-moving traveling waves occurs upon reflection at the end walls. If such reflection-induced phase shifts were to occur, then additional phase factors $e^{-in\phi_0}$ and $e^{-in\phi_L}$ would need to be included in the above expressions in order to explicitly take into account/properly mathematically describe general/generic phase shifts associated with reflection of the individual right- and left-moving plane waves at {non-perfectly rigid} end-walls of the SWT.

Is it possible to obtain an analytic, closed-form expression for the infinite series associated with the complex amplitudes $\tilde{A}(\tilde{k})$ and $\tilde{B}(\tilde{k})$? The answer is a most definite **yes!**

Defining $t \equiv 2\kappa L > 0$ and $x \equiv 2\kappa L$, and noting that $\sum_{n=0}^{\infty} e^{-nt} e^{-inx} = \sum_{n=0}^{\infty} e^{-nt} \cos nx - i \sum_{n=0}^{\infty} e^{-nt} \sin nx$ the analytic/closed-form expressions for the two ∞ series on the RHS of this relation, for $t \geq 0$ are ^[1]:

$$\sum_{n=0}^{\infty} e^{-nt} \sin nx = \frac{1}{2} \left(\frac{\sin x}{\cosh t - |\cos x|} \right) \quad \text{and:} \quad \sum_{n=0}^{\infty} e^{-nt} \cos nx = \frac{1}{2} \left(\frac{\sinh t}{\cosh t - \cos x} + 1 \right)$$

$$\text{Thus: } \boxed{\sum_{n=0}^{\infty} e^{-nt} \cos nx - i \sum_{n=0}^{\infty} e^{-nt} \sin nx = \sum_{n=0}^{\infty} e^{-nt} e^{-inx} = \frac{1}{2} \left[\left(\frac{\sinh t}{\cosh t - \cos x} + 1 \right) - i \left(\frac{\sin x}{\cosh t - |\cos x|} \right) \right]}$$

Hence the analytic/closed-form expression for complex $\tilde{A}(\tilde{k}) = \tilde{B}(\tilde{k})$ is:

$$\boxed{\tilde{A}(\tilde{k}) = \tilde{B}(\tilde{k}) = A_o \left[\sum_{n=0}^{\infty} e^{-2n\kappa L} e^{-2inkL} \right] = \frac{1}{2} A_o \left[\left(1 + \frac{\sinh(2\kappa L)}{\cosh(2\kappa L) - \cos(2kL)} \right) - i \left(\frac{\sin(2kL)}{\cosh(2\kappa L) - |\cos(2kL)|} \right) \right]}$$

Thus, the overall instantaneous complex pressure amplitude $\tilde{p}(z, t)$ is:

$$\boxed{\begin{aligned} \tilde{p}(z, t) &= \left[\tilde{A}(\tilde{k}) e^{-i\tilde{k}^* z} + \tilde{B}(\tilde{k}) e^{+i\tilde{k} z} \right] e^{+i\omega t} = \tilde{A}(\tilde{k}) \left[e^{-i\tilde{k}^* z} + e^{+i\tilde{k} z} \right] e^{+i\omega t} = 2\tilde{A}(\tilde{k}) e^{-\kappa z} \cos(kz) e^{+i\omega t} \\ &= A_o e^{-\kappa z} \cos(kz) \left[\left(1 + \frac{\sinh(2\kappa L)}{\cosh(2\kappa L) - \cos(2kL)} \right) - i \left(\frac{\sin(2kL)}{\cosh(2\kappa L) - |\cos(2kL)|} \right) \right] e^{+i\omega t} \end{aligned}}$$

Note that the physics associated with standing plane acoustic waves inside a SWT is similar to that associated with standing plane electromagnetic/visible light waves inside a Fabry-Perot etalon with semi-transparent/partially-silvered and/or dielectric-coated plane-parallel mirrors!

$$\text{Defining: } \boxed{\tilde{\mathcal{F}}(\tilde{k}) \equiv \left[\left(1 + \frac{\sinh(2\kappa L)}{\cosh(2\kappa L) - \cos(2kL)} \right) - i \left(\frac{\sin(2kL)}{\cosh(2\kappa L) - |\cos(2kL)|} \right) \right]}$$

$$\text{Then } \tilde{p}(z, t) \text{ can be written as: } \boxed{\tilde{p}(z, t) = A_o \tilde{\mathcal{F}}(\tilde{k}) e^{-\kappa z} \cos(kz) e^{+i\omega t}}$$

The 1-D complex particle velocity is of the general form $\tilde{u}_z(z, t) = \tilde{u}_z(z) e^{+i\omega t}$ and is related to the complex pressure $\tilde{p}(z, t)$ via the 1-D Euler equation:

$$\boxed{-\rho_o \frac{\partial \tilde{u}_z(z, t)}{\partial t} = \frac{\partial \tilde{p}(z, t)}{\partial z}}$$

$$\text{Thus: } \boxed{\tilde{u}_z(z, t) = -i \frac{1}{\omega \rho_o} A_o \tilde{\mathcal{F}}(\tilde{k}) e^{-\kappa z} [\kappa \cos kz + k \sin kz] e^{+i\omega t}}$$

Since $v(\omega) = \omega/k(\omega)$ this relation can also be written as:

$$\boxed{\tilde{u}_z(z, t) = -i \frac{1}{\rho_o v} A_o \tilde{\mathcal{F}}(\tilde{k}) e^{-\kappa z} \left[\left(\frac{\kappa}{k} \right) \cos kz + \sin kz \right] e^{+i\omega t}}$$

For inviscid fluid flow, note that $(\kappa/k) \ll 1$ or equivalently, that $\kappa \ll k$.

The complex longitudinal particle displacement {from its nominal equilibrium position} $\tilde{\xi}(z, t)$ is related to the complex longitudinal particle velocity $\tilde{u}_z(z, t)$ via $\tilde{u}_z(z, t) = \partial \tilde{\xi}(z, t) / \partial t$.

Thus:
$$\tilde{\xi}_z(z, t) = -\frac{1}{\omega \rho_o v} A_o \tilde{\mathcal{F}}(\tilde{k}) e^{-\kappa z} \left[\left(\frac{\kappa}{k} \right) \cos kz + \sin kz \right] e^{+i\omega t}$$

The complex specific acoustic impedance $\tilde{z}(z)$ of the SWT tube at the position z is defined as the ratio of complex pressure to complex longitudinal particle velocity:

$$\tilde{z}(z) \equiv \frac{\tilde{p}(z, t)}{\tilde{u}_z(z, t)} = \frac{\tilde{p}(z) e^{+i\omega t}}{\tilde{u}_z(z) e^{+i\omega t}} = \frac{\tilde{p}(z)}{\tilde{u}_z(z)} = +i \rho_o v \frac{\cos kz}{\left[\left(\frac{\kappa}{k} \right) \cos kz + \sin kz \right]}$$

Note that the complex specific acoustic impedance $\tilde{z}(z)$ of the SWT is purely imaginary, and is also a time-independent quantity, since $\tilde{p}(z, t)$ and $\tilde{u}_z(z, t)$ have the same time-dependence factor $e^{+i\omega t}$. The SI units of complex specific acoustic impedance $\tilde{z}(z)$ are $Pa \cdot s / m = kg / s \cdot m^2$, also known simply as acoustic ohms.

The 1-D complex longitudinal acoustic intensity $\tilde{I}_z(z)$ at the position z is defined as:

$$\tilde{I}_z(z) \equiv \frac{1}{2} \tilde{p}(z, t) \tilde{u}_z^*(z, t) = \frac{1}{2} \tilde{p}(z) \tilde{u}_z^*(z) = -i \frac{A_o^2}{2 \rho_o v_o} \left| \tilde{\mathcal{F}}(\tilde{k}) \right|^2 e^{-2\kappa z} \cos(kz) \left[\left(\frac{\kappa}{k} \right) \cos kz + \sin kz \right]$$

The complex longitudinal acoustic intensity $\tilde{I}_z(z)$ in the SWT is purely imaginary and is also a time-independent quantity. The SI units of complex acoustic intensity $\tilde{I}_z(z)$ are $Watts / m^2$.

The time-averaged total acoustic energy density $\langle e_{tot}(z) \rangle$ at the position z is the additive sum of the individual time-averaged acoustic potential energy density $\langle e_{pot}(z) \rangle$ and the time-averaged acoustic kinetic energy density $\langle e_{kin}(z) \rangle$:

$$\langle e_{tot}(z) \rangle = \langle e_{pot}(z) \rangle + \langle e_{kin}(z) \rangle \equiv \frac{1}{4} \frac{|\tilde{p}(z, t)|^2}{\rho_o v^2} + \frac{1}{4} \rho_o |\tilde{u}_z(z, t)|^2 = \frac{1}{4} \frac{|\tilde{p}(z)|^2}{\rho_o v^2} + \frac{1}{4} \rho_o |\tilde{u}_z(z)|^2$$

The time-averaged energy densities are purely real, time-independent quantities. The SI units of energy density are $Joules / m^3$.

Precisely at one of the resonant frequencies of the SWT $f_n = v/\lambda_n = nv/2L$, with both of the p and u_z mics located e.g. at $z = L$, then $k_n L = 2\pi L/\lambda_n = n\pi$, and hence $\cos(k_n L) = \cos(n\pi) = (-1)^n$, $\cos(2k_n L) = \cos(2n\pi) = 1$, $\sin(k_n L) = \sin(n\pi) = 0$ and $\sin(2k_n L) = \sin(2n\pi) = 0$ and thus the instantaneous overall complex pressure amplitude at $z = L$ on the n^{th} resonance of the SWT becomes:

$$\tilde{p}_n(z = L, t) = (-1)^n A_o e^{-\kappa L} \left(\frac{\sinh(2\kappa L)}{\cosh(2\kappa L) - 1} + 1 \right) e^{+i\omega_n t}$$

Now since $\coth x = \frac{1}{\tanh x} = \frac{\cosh x}{\sinh x}$ and using the fact that $\tanh\left(\frac{1}{2}x\right) = \frac{\cosh x - 1}{\sinh x}$, then we see that

$\coth\left(\frac{1}{2}x\right) = \frac{1}{\tanh\left(\frac{1}{2}x\right)} = \frac{\sinh x}{\cosh x - 1}$ and thus if $x = 2\kappa L$ we see that the above expression for

$\tilde{p}_n(z = L, t)$ on the SWT resonances can equivalently be written as:

$$\tilde{p}_n(z = L, t) = (-1)^n A_o e^{-\kappa L} [\coth(\kappa L) + 1] e^{+i\omega_n t}$$

Thus, we see that there are pressure anti-nodes at both $z = 0$ and $z = L$ on the resonances of the SWT for infinitely rigid/closed end walls.

The instantaneous 1-D longitudinal particle velocity at $z = L$ on the resonances of the SWT is:

$$\tilde{u}_{z_n}(z = L, t) = -i \frac{(-1)^n}{\rho_o v} A_o e^{-\kappa L} \left(\frac{\kappa}{k_n} \right) \left(\frac{\sinh(2\kappa L)}{\cosh(2\kappa L) - 1} + 1 \right) e^{+i\omega_n t}$$

Again, using the relation $\coth\left(\frac{1}{2}x\right) = \frac{1}{\tanh\left(\frac{1}{2}x\right)} = \frac{\sinh x}{\cosh x - 1}$ the longitudinal particle velocity at $z = L$ on one of the resonances of the SWT can be rewritten as:

$$\tilde{u}_{z_n}(z = L, t) = -i \frac{(-1)^n}{\rho_o v} A_o e^{-\kappa L} \left(\frac{\kappa}{k_n} \right) [\coth(\kappa L) + 1] e^{+i\omega_n t}$$

Note that on a resonance of the SWT, $\tilde{u}_{z_n}(z = L, t)$ is -90° out-of-phase relative to $\tilde{p}_n(z = L, t)$.

The instantaneous 1-D longitudinal particle displacement at $z = L$ on a resonance of the SWT is:

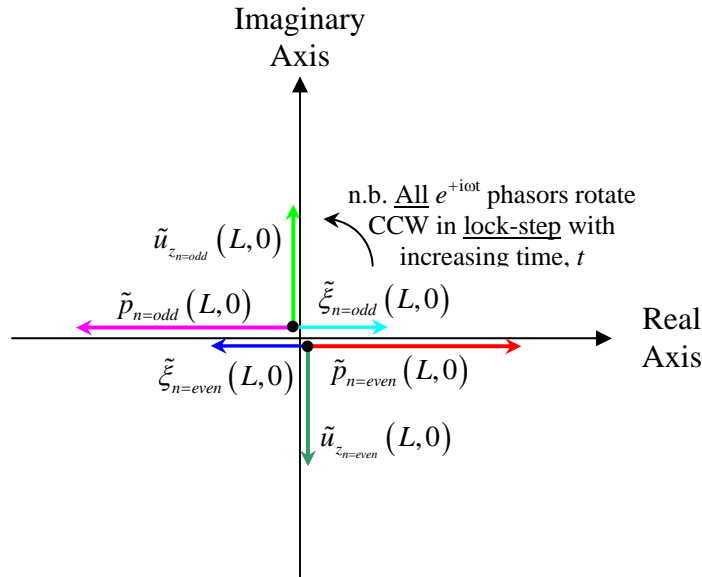
$$\tilde{\xi}_{z_n}(z = L, t) = -\frac{(-1)^n}{\omega_n \rho_o v} A_o e^{-\kappa L} \left(\frac{\kappa}{k_n} \right) \left\{ \left(\frac{\sinh(2\kappa L)}{\cosh(2\kappa L) - 1} + 1 \right) \right\} e^{+i\omega_n t}$$

Again, using the relation $\coth\left(\frac{1}{2}x\right) = \frac{1}{\tanh\left(\frac{1}{2}x\right)} = \frac{\sinh x}{\cosh x - 1}$ the longitudinal particle displacement at $z = L$ on one of the resonances of the SWT can be rewritten as:

$$\xi_{z_n}^{\sim}(z = L, t) = -\frac{(-1)^n}{\omega_n \rho_o} A_o e^{-\kappa L} \left(\frac{\kappa}{k_n}\right) (\coth(\kappa L) + 1) e^{+i\omega_n t}$$

Thus we see that on the resonances of the SWT, $\xi_{z_n}^{\sim}(z = L, t)$ is -90° out of phase relative to $\tilde{u}_{z_n}(z = L, t)$ and is -180° out of phase relative to $\tilde{p}_n(z = L, t)$.

In terms of a phasor diagram, the complex pressure $\tilde{p}_n(z = L, t)$, complex longitudinal particle velocity $\tilde{u}_{z_n}(z = L, t)$ and complex longitudinal displacement $\xi_{z_n}^{\sim}(z = L, t)$ on the resonances of the SWT as observed at $z = L$ and at time $t = 0$ are oriented as shown in the figure below:



The complex specific acoustic impedance at $z = L$ on the resonances of the SWT is purely imaginary and time-independent:

$$z_n(z = L) \equiv \frac{\tilde{p}_n(z = L, t)}{\tilde{u}_{z_n}(z = L, t)} = +i \rho_o v \left(\frac{k_n}{\kappa}\right)$$

The complex longitudinal intensity at $z = L$ on the resonances of the SWT is also purely imaginary and time-independent:

$$\tilde{I}_{z_n}(z = L) \equiv \frac{1}{2} \tilde{p}_n(z = L) \tilde{u}_{z_n}^*(z = L) = +i \frac{A_o^2}{2\rho_o v} e^{-2\kappa L} \left(\frac{\kappa}{k_n}\right) [\coth(\kappa L) + 1]^2$$

Note that since there are acoustic *standing* waves present in the SWT, the complex longitudinal sound intensity $\tilde{I}_z(z)$ must be purely reactive (i.e. purely imaginary). Any non-zero value associated with the real component of $\tilde{I}_z(z)$ is due to an actual {time-averaged} flux, or flow of energy down the SWT – this cannot be due to a *standing* wave – only a *traveling* sound wave can have this!

The *time-averaged* total energy density at the position $z = L$ on a resonance of the SWT is:

$$\begin{aligned} \langle e_{n\text{tot}}(z=L) \rangle &\equiv \langle e_{n\text{pot}}(z=L) \rangle + \langle e_{n\text{kin}}(z=L) \rangle = \frac{1}{4} \frac{|\tilde{p}_n(z=L, t)|^2}{\rho_o v^2} + \frac{1}{4} \rho_o |\tilde{u}_{z_n}(z=L, t)|^2 \\ &= \frac{1}{4} \frac{A_o^2}{\rho_o v^2} e^{-2\kappa L} [\coth(\kappa L) + 1]^2 + \frac{1}{4} \frac{A_o^2}{\rho_o v^2} e^{-2\kappa L} \left(\frac{\kappa}{k_n} \right)^2 [\coth(\kappa L) + 1]^2 \end{aligned}$$

Or:

$$\langle e_{n\text{tot}}(z=L) \rangle \equiv \langle e_{n\text{pot}}(z=L) \rangle + \langle e_{n\text{kin}}(z=L) \rangle = \frac{1}{4} \frac{A_o^2}{\rho_o v^2} e^{-2\kappa L} \left[1 + \left(\frac{\kappa}{k_n} \right)^2 \right] [\coth(\kappa L) + 1]^2$$

Note that since $(\kappa/k_n) \ll 1$ for inviscid fluid flow, we see that $\langle e_{n\text{pot}}(z=L) \rangle \gg \langle e_{n\text{kin}}(z=L) \rangle$ on the resonances of the SWT.

References:

[1] *Table of Integrals, Series and Products*, I.S. Gradshteyn and I.M. Ryzhik, page 42, Academic Press, Inc. 1980.

In this reference, please note that the expression $\sum_{n=0}^{\infty} e^{-nt} \sin nx = \left(\frac{\sin x}{\cosh t - \cos x} \right)$ is factually in error in the 2nd and 3rd quadrants (where $\cos x < 0$). The correct expression, valid in all four quadrants is:

$$\sum_{n=0}^{\infty} e^{-nt} \sin nx = \left(\frac{\sin x}{\cosh t - |\cos x|} \right)$$

[2] *ibid*, page 25.