Physics 498 Physics of Music/Musical Instruments

Lecture II

Simple One-Dimensional Vibrating Systems

One method of producing a sound relies on a physical object (e.g. various types of musical instruments – stringed and wind instruments in particular) to be made to vibrate, by whatever means possible. This vibration is (clearly) mechanical in nature.

Mechanical vibration explicitly means a *displacement* of the (at least some portions of the) matter/material the object is comprised of *from its equilibrium position/configuration* – which requires the input of *energy* to the object in order to accomplish this – initially in the form of (static) *potential* energy (P.E.) , which as time progresses, is subsequently transformed into *kinetic* (motional) energy (K.E.). As time progresses further, the energy oscillates back and forth between potential and kinetic energy, the total energy,
E_{tot} = P.E. + K.E. remaining *constant* in time, if no energy losses (energy dissipation processes) are present in the mechanical system.

The mechanically vibrating object couples to the air surrounding it, and sound waves in the air are created, which propagate outwards from the source (the vibrating object) to an observer's ear(s). Thus a sound is heard (perceived). Thus, by *energy conservation*, some of the initial energy input to the mechanically vibrating system *is* radiated away in the form of sound energy. Eventually the mechanically vibrating system ceases to do so, because of this, and other (frictional) dissipative energy loss mechanisms present.

A very simple example of a vibrating system is a mass on a spring (a crude model of a vibrating musical instrument) which undergoes so-called 1-D simple harmonic motion:

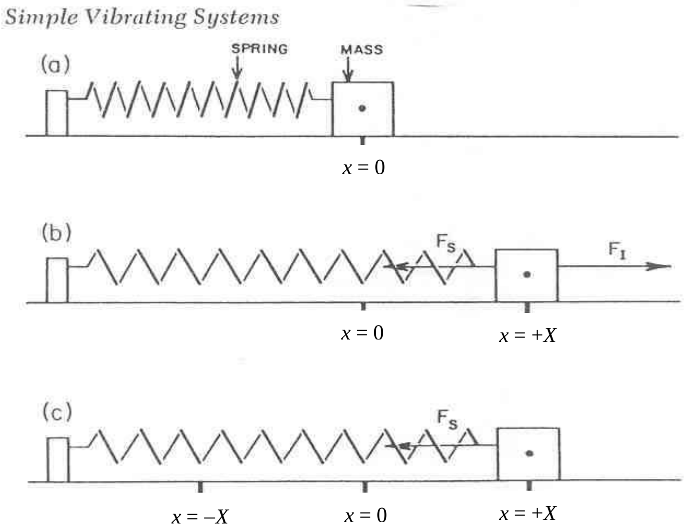


FIG. 1. A mass attached to a spring and resting on a smooth table. (a) Equilibrium position. (b) Mass displaced and held in new position. (c) Mass released.

If the mass M is horizontally displaced from its equilibrium ($x = 0$) position by pulling on it to the right, as shown in the above figure the force necessary to accomplish this is $\boxed{F_I = +kX}$, where k = the so-called "spring constant" of the spring (k has metric units of Newtons/meter) and X = the initial displacement of the mass M from its $x = 0$ equilibrium position.

At time $t = 0$ the mass is released. At that instant, the only {horizontal} force acting on the mass is due to the restoring force of the spring: $\boxed{F_S(t=0) = -kX = -F_I}$. However, from Newton's 2nd Law $F = Ma$, and therefore at time $t = 0$: $\boxed{F_S(t=0) = -kX = Ma(t=0)}$.

As time progresses the mass M oscillates horizontally back and forth about its $x = 0$ equilibrium position, exhibiting sinusoidal/harmonic motion. Mathematically, the time-dependence of this horizontal sinusoidal/harmonic motion is described by:

Longitudinal displacement from equilibrium: $\boxed{x(t) = X\cos(2\pi f t) = X\cos(\omega t)}$ (m)

(meters) displacement amplitude (meters) frequency of oscillation (cycles per second = Hertz)
cps Hz

Omega: $\boxed{\omega \equiv 2\pi f = \text{angular frequency (units = radians per second)}}$

Period of oscillation: $\boxed{\tau \equiv \frac{1}{f} = \frac{2\pi}{\omega}}$ (seconds)

The instantaneous horizontal speed of the moving mass $v(t)$ with time t is defined as the time rate of change of the horizontal position (longitudinal displacement) of the moving mass with time t, physically, $v(t)$ is the instantaneous local slope of the $x(t)$ vs. t graph at time t:

$$v(t) = \frac{\Delta x(t)}{\Delta t} = \frac{dx(t)}{dt} = \text{total derivative of } x \text{ with respect to time, } t \text{ \{since 1-D partial => total\}.}$$

$$v(t) = \frac{d}{dt}\left(x(t)\right) = \frac{d}{dt}\left[X\cos(2\pi f t)\right] = -2\pi f X\sin(2\pi f t) = -\omega X\sin(\omega t) = -V\sin(\omega t)$$

We see that: $\boxed{V = \omega X = 2\pi f X}$

i.e. the speed "amplitude", V = max speed is related to the displacement amplitude, X by this formula

Instantaneous Horizontal Speed of the Moving Mass: $\boxed{v(t) = -V\sin(2\pi f t) = -V\sin(\omega t)}$ (m/s)

(meters/sec) speed amplitude (m/s) frequency of oscillation (cycles per second = Hertz)

The instantaneous horizontal acceleration of the moving mass $a(t)$ with time t is defined as the time rate of change of the horizontal speed of the moving mass with time t, physically, $a(t)$ is the instantaneous local slope of the $v(t)$ vs. t graph at time t:

$$a(t) = \frac{\Delta v(t)}{\Delta t} = \frac{dv(t)}{dt} = \text{total derivative of } v(t) \text{ with respect to time, } t.$$

$$a(t) = \frac{d}{dt}\left(v(t)\right) = \frac{d}{dt}\left[-V\sin\left(2\pi ft\right)\right] = -2\pi fV\cos\left(2\pi ft\right) = -\omega V\cos\left(\omega t\right) = -A\cos\left(\omega t\right)$$

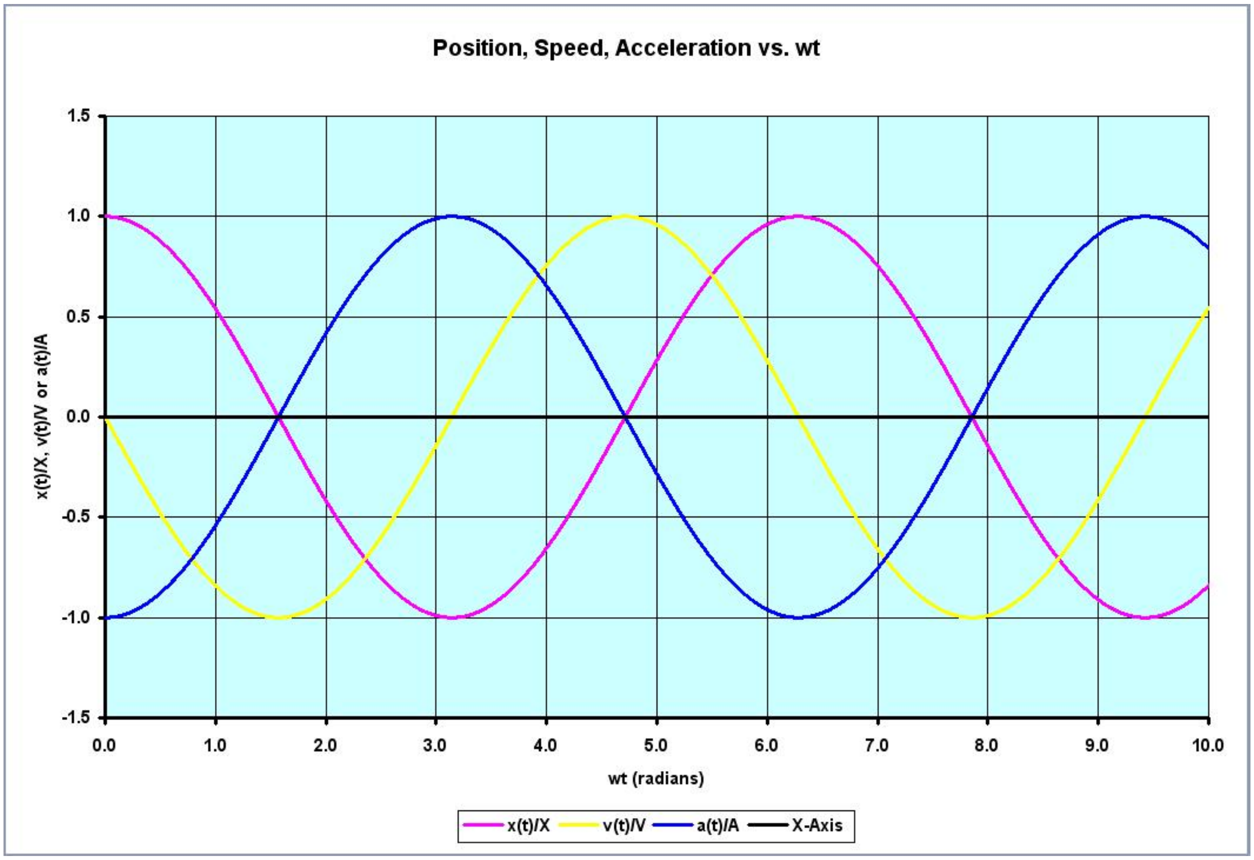
<u>We see that</u>: $\boxed{A = \omega V = 2\pi fV}$ <u>but</u>: $\boxed{V = \omega X = 2\pi fX}$ $\therefore$ $\boxed{A = \omega^2 X = \left(2\pi f\right)^2 X}$

i.e. the acceleration "amplitude", A = max acceleration is related to the displacement amplitude, X by this formula

<u>Instantaneous Horizontal Accel. of the Moving Mass</u>: $\boxed{a(t) = -A\cos\left(2\pi ft\right) = -A\cos\left(\omega t\right)}$ $\left(m/s^2\right)$

(meters/sec^2) acceleration <u>amplitude</u> (m/s^2) frequency of oscillation (cycles per second = Hertz)

The time dependence of the longitudinal position, *x(t)* (i.e. displacement of the mass from its equilibrium position) vs. time, *t* and longitudinal speed of the mass, *v(t)* vs. time, *t* and longitudinal acceleration *a(t)* vs. time, *t* are shown in the figure below; note that each has been normalized to their respective amplitudes (note the phase relation between $x(t)$, $v(t)$ and $a(t)$):



Once the mass M has been set in motion, Newton's 2nd Law tells us: $\boxed{F(t) = -kx(t) = Ma(t)}$

But: $\boxed{x(t) = X\cos(2\pi ft) = X\cos(\omega t)}$ and: $\boxed{a(t) = -A\cos(2\pi ft) = -A\cos(\omega t)}$

However, from above, we also know that: $\boxed{A = \omega^2 X = (2\pi f)^2 X}$ $\therefore$ $\boxed{-kX = -\omega^2 MX}$

Thus, the frequency f and angular frequency ω of oscillation of the mass M on the spring are:

$\boxed{f = \frac{1}{2\pi}\sqrt{\frac{k}{M}}}$ Cycles per second, or Hz and $\boxed{\omega = 2\pi f = \sqrt{\frac{k}{M}}}$ (radians/sec)

The period of oscillation τ of the mass M on the spring is: $\boxed{\tau = \frac{1}{f} = 2\pi\sqrt{\frac{M}{k}}}$ (seconds)

The instantaneous <u>potential</u> energy <u>*stored*</u> in the stretched/compressed spring is:

$\boxed{P.E.(t) = \frac{1}{2}kx^2(t)}$ (Joules)

The instantaneous <u>kinetic</u> energy associated with the <u>*moving*</u> mass, M is:

$\boxed{K.E.(t) = \frac{1}{2}Mv^2(t)}$ (Joules)

The potential energy of the spring and the kinetic energy of the moving mass are both time dependent:

$$\boxed{\begin{aligned} P.E.(t) &= \frac{1}{2}kx^2(t) = \frac{1}{2}kA^2\cos^2(\omega t) \\ K.E.(t) &= \frac{1}{2}Mv^2(t) = \frac{1}{2}MV^2\sin^2(\omega t) \end{aligned}}$$

<u>However</u>: $\boxed{V = \omega X}$ and: $\boxed{\omega = \sqrt{\frac{k}{M}}}$ or: $\boxed{k = M\omega^2}$

<u>Thus</u>:

$$P.E.(t) = \frac{1}{2}kx^2(t) = \frac{1}{2}kX^2\cos^2(\omega t)$$

$$K.E.(t) = \frac{1}{2}Mv^2(t) = \frac{1}{2}MV^2\sin^2(\omega t) = \frac{1}{2}M\omega^2X^2\sin^2(\omega t) = \frac{1}{2}kX^2\sin^2(\omega t)$$

Let us define: $\boxed{E_o \equiv \frac{1}{2}kX^2 = \frac{1}{2}M\omega^2X^2}$

Then: $$\boxed{\begin{aligned} P.E.(t) &= E_o \cos^2(\omega t) \\ K.E.(t) &= E_o \sin^2(\omega t) \end{aligned}}$$

We define the *total* energy, E_{Tot} as:

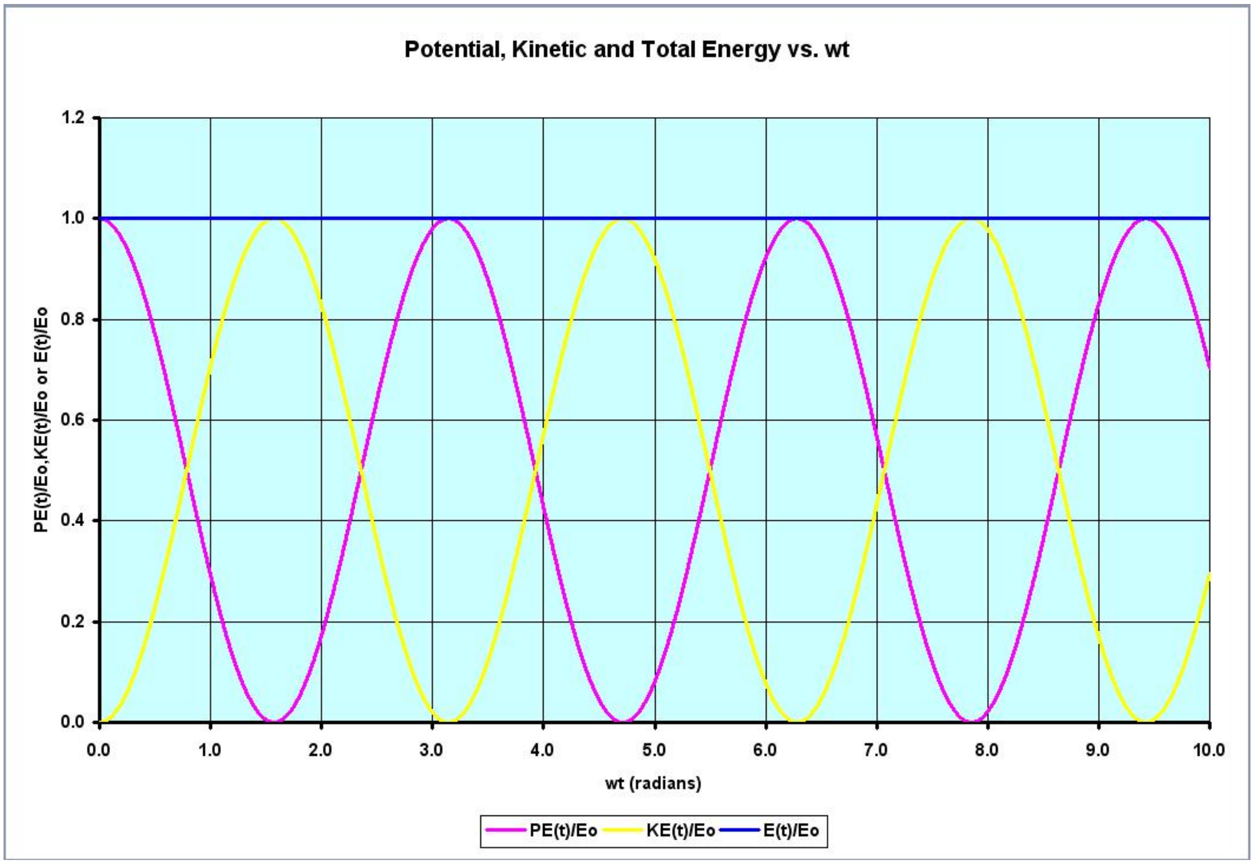
$$\boxed{E_{Tot}(t) = P.E.(t) + K.E.(t) = E_o \cos^2(\omega t) + E_o \sin^2(\omega t) = E_o \left\{\cos^2(\omega t) + \sin^2(\omega t)\right\}}$$

Using the trigonometric identity $1 = \cos^2 x + \sin^2 x$ we see that:

$$\boxed{E_{Tot}(t) = E_o = \frac{1}{2} k X^2 = \frac{1}{2} M \omega^2 X^2} = \text{constant!}$$

The total energy in (spring + mass) system *is* constant — due to conservation of energy!!

Graphs of $P.E.(t)$, $K.E.(t)$, and $E_{Tot}(t)$ vs. time (all normalized to E_o):



Note that the $P.E.(t)$, $K.E.(t)$, and $E_{Tot}(t)$ are all always > 0 (i.e. never negative)!!!

Note further that energy/energies are additive, scalar quantities.

A <u>real</u> vibrating spring — mass system suffers from various <u>energy</u> <u>loss</u> <u>mechanisms</u>:
- — friction – mass slides on surface, mass also slides through viscous air
- — spring also dissipates energy internally each time it is flexed (another type of friction)

— Thus, motion of mass on a spring is damped by frictional processes.

— Original/initial energy, $E_{Tot} = E_o$ is dissipated by frictional processes.

— Initial energy E_o ultimately winds up as heat (another form of energy) thus the mass, spring, surface and air all heat up with time…

Damping / dissipation affects the vibrational motion as time progresses:

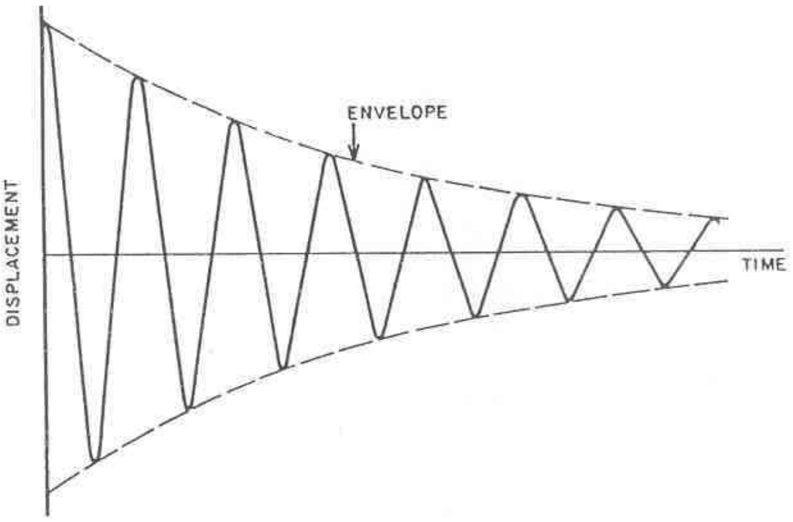


FIG. 3. Graph of displacement versus time for a damped vibration.

Dissipative processes tend to <u>lower</u> the frequency of oscillation of a vibrating system. Small damping – slight decrease in the oscillation frequency. Heavy damping – no oscillation(s) at all!

More realistic motion of a vibrating mass on spring — e.g. driving it with periodic force
- have to get mass moving first (initially at rest), takes a while for oscillations to build up
- reaches a <u>steady state</u> displacement amplitude
- switch off the driving force, displacement amplitude decays away

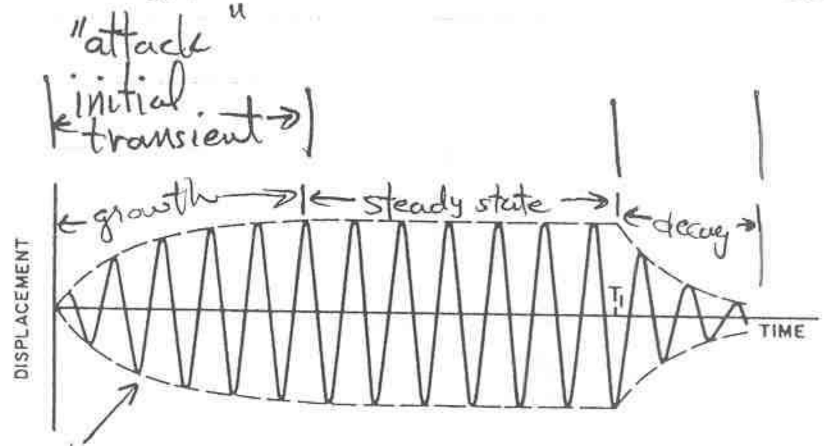


FIG. 4. Growth and decay of a vibration.

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