Examples of Complex Sound Fields (Continued):

Example # 3: Point Monopole Sound Source – Spherical Waves Propagating in "Free Air":

Note: no electromagnetic analog exists for this acoustic example, due to the manifest *vectorial* nature of the *EM* field – mediated at the microscopic level by the spin-1 photon. The so-called electric monopole {E(0)} and/or magnetic monopole {M(0)} radiation associated e.g. with a spherically-symmetric, radially oscillating electric charge distribution $\rho_e(\vec{r},t) = q_o\delta^3(\vec{r})e^{i\omega t}$ and/or magnetic charge distribution $\rho_m(\vec{r},t) = g_o\delta^3(\vec{r})e^{i\omega t}$ ***cannot*** occur.

Imagine a spherically-symmetric, point sound source located at the origin of coordinates that emits monochromatic spherical waves into "free air". In general, the instantaneous pressure $p(\vec{r},t)$ at the space-time point $(\vec{r},t)$ {n.b. for $r > 0$!!!} satisfies the homogeneous 3-D wave equation:

$$\nabla^2 p(\vec{r},t) - \frac{1}{c^2}\frac{\partial^2 p(\vec{r},t)}{\partial t^2} = 0$$

The gradient $\vec{\nabla}$ and Laplacian ∇^2 operators in 3-D spherical-polar (r,θ,φ) coordinates are:

$$\vec{\nabla} = \frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\hat{\varphi}$$

and:

$$\nabla^2 \equiv \vec{\nabla}\bullet\vec{\nabla} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

A spherically-symmetric time-dependent scalar pressure field has rotational invariance/ rotational symmetry and therefore can have no explicit θ and/or φ-dependence – only *r*-dependence. Thus $p(\vec{r},t) = p(r,t) \neq fcn(\theta,\varphi)$, and hence for a spherically-symmetric sound source located at the origin of coordinates, the 3-D wave equation for the scalar pressure field associated with this sound source becomes:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial p(\vec{r},t)}{\partial r}\right) - \frac{1}{c^2}\frac{\partial^2 p(\vec{r},t)}{\partial t^2} = 0$$

$$\frac{\partial^2 p(\vec{r},t)}{\partial r^2} + \frac{2}{r}\frac{\partial p(\vec{r},t)}{\partial r} - \frac{1}{c^2}\frac{\partial^2 p(\vec{r},t)}{\partial t^2} = 0$$

The pressure wave solution associated with a spherically-symmetric monochromatic sound source is a purely ***real*** quantity - a spherical-outgoing harmonic wave of the form:

$$p(r,t) = \frac{B_o}{r}\cos(\omega t - kr) \quad (Pascals).$$

Note that the constant B_o has *SI* units of $Pascal\text{-}m$.

The particle velocity $\vec{u}(\vec{r},t)$ associated with this problem is {again} determined via use of Euler's equation. Since $p(\vec{r},t)=p(r,t)\neq fcn(\theta,\varphi)$, $\vec{\nabla}p(\vec{r},t)=\vec{\nabla}p(r,t)=\left(\partial p(r,t)/\partial r\right)\hat{r}$ and thus the particle velocity is only in the radial direction, i.e. $\vec{u}(\vec{r},t)=u_{\mathrm{r}}(r,t)\hat{r}\neq fcn(\theta,\varphi)$:

$$\frac{\partial\vec{u}(\vec{r},t)}{\partial t}=-\frac{1}{\rho_o}\vec{\nabla}p(\vec{r},t)\;\Rightarrow\;\frac{\partial u_{\mathrm{r}}(r,t)}{\partial t}\hat{r}=-\frac{1}{\rho_o}\frac{\partial p(r,t)}{\partial r}\hat{r}\;\Rightarrow\;\frac{\partial u_{\mathrm{r}}(r,t)}{\partial t}=-\frac{1}{\rho_o}\frac{\partial p(r,t)}{\partial r}$$

The solution for the 1-D radial particle velocity associated with this problem is also a purely <u>*real*</u> quantity, and is of the form:

$$u_{\mathrm{r}}(r,t)=\frac{B_o k}{\omega\rho_o r}\left\{\cos(\omega t-kr)+\frac{1}{kr}\sin(\omega t-kr)\right\}\;\;(m/s)$$

Using $c=\omega/k$ and $z_o\equiv\rho_o c$ this becomes:

$$u_{\mathrm{r}}(r,t)=\frac{B_o}{z_o r}\left\{\cos(\omega t-kr)+\frac{1}{kr}\sin(\omega t-kr)\right\}\;\;(m/s)$$

We can now "complexify" the radial-outgoing spherical pressure and particle velocity waves:

$$p(r,t)=\frac{B_o}{r}\cos(\omega t-kr)\;\Rightarrow\;\tilde{p}(r,t)=\frac{\tilde{B}}{r}e^{i(\omega t-kr)}$$

$$u_{\mathrm{r}}(r,t)=\frac{1}{z_o}\frac{B_o}{r}\left\{\cos(\omega t-kr)+\frac{1}{kr}\sin(\omega t-kr)\right\}\;\Rightarrow\;\tilde{u}_{\mathrm{r}}(r,t)=\frac{1}{z_o}\frac{\tilde{B}}{r}\left[1-\frac{i}{kr}\right]e^{i(\omega t-kr)}$$

The "amplitude" $\tilde{B}=\left|\tilde{B}\right|e^{i\varphi_B}$ is now (in general) complex, in order to accommodate an (arbitrary) phase. The <u>magnitudes</u> of the complex pressure and 1-D radial particle velocity are:

$$\left|\tilde{p}(r,t)\right|=\frac{\left|\tilde{B}\right|}{r}\;\;\text{and:}\;\;\left|\tilde{u}_{\mathrm{r}}(r,t)\right|=\frac{1}{z_o}\frac{\left|\tilde{B}\right|}{r}\sqrt{1+(1/kr)^2}$$

The phases of the complex pressure and 1-D radial particle velocity are:

$$\varphi_p=\varphi_B\;\;\text{and:}\;\;\varphi_u=\tan^{-1}(-1/kr)+\varphi_B=-\tan^{-1}(1/kr)+\varphi_B$$

Thus we also see that:

$$\Delta\varphi_{p-u}\equiv\varphi_p-\varphi_u=\varphi_B-\left(-\tan^{-1}(1/kr)+\varphi_B\right)=+\tan^{-1}(1/kr)$$

The complex sound field $\tilde{S}(\vec{r},t)$ associated with a "point" monopole sound source radiating monochromatic, radially-outgoing spherical waves has ***three*** basic regions, or zones:

(a) **The "near" zone:** $kr \ll 1$, the 1-D radial particle velocity $\tilde{u}_{\mathrm{r}}(r) \sim -i\left|\tilde{B}\right|/z_o kr^2$ is largely ***imaginary*** (i.e. ***reactive***), decreasing as $\sim 1/r^2$ (while the pressure $\tilde{p}(r,t) \sim \left|\tilde{B}\right|/r$ decreases as $\sim 1/r$), and where the particle velocity *lags* the pressure by $\varphi_p - \varphi_u = \tan^{-1}(1/kr) \sim \tan^{-1}(\infty) \sim 90^{\mathrm{o}}$ in phase. This region of the complex sound field $\tilde{S}(\vec{r},t)$ is ***reactive***, largely consisting of ***non-propagating*** acoustic energy, and is also ***inertia***-like (i.e. ***mass***-like), because $u_{\mathrm{r}i}(r) = \mathrm{Im}\{\tilde{u}_{\mathrm{r}}(r)\} < 0$ for $kr \ll 1$.

(b) **The "intermediate" zone:** $kr \sim 1$, the 1-D radial particle velocity $\tilde{u}_{\mathrm{r}}(r) \sim (1-i)\left|\tilde{B}\right|/2z_o r$ has ~ comparable ***real*** and ***imaginary*** components, decreasing somewhat/slightly faster than $\sim 1/r$ (while the pressure $\tilde{p}(r,t) \sim \left|\tilde{B}\right|/r$ decreases as $\sim 1/r$), and where the particle velocity *lags* the pressure by $\varphi_p - \varphi_u = \tan^{-1}(1/kr) \sim \tan^{-1}(1) \sim 45^{\mathrm{o}}$ in phase. This region of the complex sound field $\tilde{S}(\vec{r},t)$ is ~ an ***equal*** mix of ***propagating*** and ***non-propagating*** acoustic energy.

(c) **The "far" (or radiation) zone:** $kr \gg 1$, the 1-D radial particle velocity $\tilde{u}_{\mathrm{r}}(r) \sim 1\left|\tilde{B}\right|/z_o r$ is largely ***real*** (i.e. ***active***), decreasing $\sim 1/r$ (while the pressure $\tilde{p}(r,t) \sim \left|\tilde{B}\right|/r$ also decreases as $\sim 1/r$), and where the 1-D radial particle velocity is ***in-phase*** with the pressure, i.e. $\varphi_p - \varphi_u = \tan^{-1}(1/kr) \sim \tan^{-1}(0) \sim 0^{\mathrm{o}}$. This region of the complex sound field $\tilde{S}(\vec{r},t)$ is ***active***, dominated by ***propagating*** acoustic energy.

The 1-D radial complex *specific* acoustic impedance and its magnitude are:

$$\tilde{z}_{\mathrm{r}}(r) \equiv \frac{\tilde{p}(r,t)}{\tilde{u}_{\mathrm{r}}(r,t)} = z_o \frac{1}{\left[1 - i/kr\right]} = z_o \frac{\left[1 + i/kr\right]}{\left[1 + (1/kr)^2\right]} \quad \text{and:} \quad \left|\tilde{z}_{\mathrm{r}}(r)\right| = z_o \frac{\sqrt{1 + (1/kr)^2}}{\left[1 + (1/kr)^2\right]} = z_o \frac{1}{\sqrt{1 + (1/kr)^2}}$$

The real and imaginary parts of the 1-D radial complex *specific* acoustic impedance are thus:

$$z_{\mathrm{r}r}(r) = \mathrm{Re}\{\tilde{z}_{\mathrm{r}}(r)\} = z_o \frac{1}{\left[1 + (1/kr)^2\right]} \quad \text{and:} \quad z_{\mathrm{r}i}(r) = \mathrm{Im}\{\tilde{z}_{\mathrm{r}}(r)\} = z_o \frac{1/kr}{\left[1 + (1/kr)^2\right]}$$

The phase of the 1-D radial complex *specific* acoustic impedance is:

$$\varphi_z = \Delta\varphi_{p-u} = \varphi_p - \varphi_u = \tan^{-1}(1/kr)$$

(a) **In the "near" zone:** $kr \ll 1$, the 1-D radial complex *specific* acoustic impedance is largely ***imaginary*** (i.e. ***reactive***): $\tilde{z}_{\mathrm{r}}(r) = z_o\left[1 + i/kr\right]\Big/\left[1 + (1/kr)^2\right] \sim i \cdot z_o kr$, increasing ~ linearly with *r*, with magnitude $\left|\tilde{z}_{\mathrm{r}}(r)\right| = z_o\Big/\sqrt{1 + (1/kr)^2} \sim z_o kr \ll z_o \simeq 413\ \Omega_a\,(@\,NTP)$ and phase $\varphi_z = \varphi_p - \varphi_u = \tan^{-1}(1/kr) \sim \tan^{-1}(\infty) \sim 90^{\mathrm{o}}$. Again, the complex sound field $\tilde{S}(\vec{r},t)$ in this region is ***inertia***-like (i.e. ***mass***-like), because $z_{\mathrm{r}i}(r) = \mathrm{Im}\{\tilde{z}_{\mathrm{r}}(r)\} > 0$ for $kr \ll 1$.

(b) **In the "intermediate zone:** $kr \sim 1$, the 1-D radial complex *specific* acoustic impedance is ~ an equal mix of ***active*** (i.e. real) and ***reactive*** (i.e. imaginary) components: $\tilde{z}_{\mathrm{r}}(r) = z_o\left[1 + i/kr\right]\Big/\left[1 + (1/kr)^2\right] \sim z_o(1+i)/2$ (~ independent of *r*) with magnitude $\left|\tilde{z}_{\mathrm{r}}(r)\right| = z_o\Big/\sqrt{1 + (1/kr)^2} \sim z_o\big/\sqrt{2}$ and phase $\varphi_z = \varphi_p - \varphi_u = \tan^{-1}(1/kr) \sim \tan^{-1}(1) \sim 45^{\mathrm{o}}$.

(c) **In the "far" (or radiation) zone:** $kr \gg 1$, the 1-D radial complex *specific* acoustic impedance is largely ***real*** (i.e. ***active***): $\tilde{z}_{\mathrm{r}}(r) = z_o\Big/\left[1 + (1/kr)^2\right] \sim z_o$ with magnitude $\left|\tilde{z}_{\mathrm{r}}(r)\right| = z_o\Big/\sqrt{1 + (1/kr)^2} \sim z_o \simeq 413\ \Omega_a\,(@\,NTP)$ (~ nearly independent of *r*) and phase $\varphi_z = \varphi_p - \varphi_u = \tan^{-1}(1/kr) \sim \tan^{-1}(0) \sim 0^{\mathrm{o}}$.

The ***time-averaged*** 1-D radial complex acoustic intensity associated with the monopole source points outward in the radial ($\hat{r}$) direction; it and its magnitude are:

$$\left\langle \tilde{I}_{\mathrm{r}}(r)\right\rangle_t \equiv \frac{1}{2}\left\langle \tilde{p}(r,t)\tilde{u}_{\mathrm{r}}^*(r,t)\right\rangle_t = \frac{1}{2}\frac{1}{z_o}\frac{|\tilde{B}|^2}{r^2}\left[1 + \frac{i}{kr}\right] \quad \text{and:} \quad \left\langle \left|\tilde{I}_{\mathrm{r}}(r)\right|\right\rangle_t = \frac{1}{2}\frac{1}{z_o}\frac{|\tilde{B}|^2}{r^2}\sqrt{1 + (1/kr)^2}$$

The real and imaginary parts of the ***time-averaged*** 1-D radial complex acoustic intensity are:

$$\left\langle I_{\mathrm{r}r}(r)\right\rangle_t = \mathrm{Re}\left\{\left\langle \tilde{I}_{\mathrm{r}}(r)\right\rangle_t\right\} = \frac{1}{2}\frac{1}{z_o}\frac{|\tilde{B}|^2}{r^2} \quad \text{and:} \quad \left\langle I_{\mathrm{r}i}(r)\right\rangle_t = \mathrm{Im}\left\{\left\langle \tilde{I}_{\mathrm{r}}(r)\right\rangle_t\right\} = \frac{1}{2}\frac{1}{z_o}\frac{|\tilde{B}|^2}{r^2}\frac{1}{kr}$$

The phase of the ***time-averaged*** 1-D radial complex acoustic intensity is:

$$\varphi_I = \varphi_z = \Delta\varphi_{p-u} = \varphi_p - \varphi_u = \tan^{-1}(1/kr)$$

(a) **In the "near" zone:** $kr \ll 1$, the ***time-averaged*** 1-D radial complex acoustic intensity is largely ***imaginary*** (i.e. ***reactive***): $\left\langle \tilde{I}_{\mathrm{r}}(r)\right\rangle_t \sim i|\tilde{B}|^2\big/2z_o kr^3$, with magnitude $\left\langle \left|\tilde{I}_{\mathrm{r}}(r)\right|\right\rangle_t \sim |\tilde{B}|^2\big/2z_o kr^3$ (decreasing as $\sim 1/r^3$) and phase $\varphi_I = \varphi_z = \varphi_p - \varphi_u = \tan^{-1}(1/kr) \sim \tan^{-1}(\infty) \sim 90^{\mathrm{o}}$. Again, the complex sound field $\tilde{S}(\vec{r},t)$ in this region is ***inertia***-like (i.e. ***mass***-like), because $I_{\mathrm{r}i}(r) = \mathrm{Im}\{\tilde{I}_{\mathrm{r}}(r)\} > 0$ for $kr \ll 1$.

(b) **In the "intermediate zone:** $kr \sim 1$, the ***time-averaged*** 1-D radial complex acoustic intensity is ~ an equal mix of ***active*** (i.e. real) and ***reactive*** (i.e. imaginary) components: $\langle \tilde{I}_{\mathrm{r}}(r) \rangle_t \sim (1+i)\left|\tilde{B}\right|^2 / 4z_o r^2$, with magnitude $\langle |\tilde{I}_{\mathrm{r}}(r)| \rangle_t \sim \left|\tilde{B}\right|^2 / 4z_o r^2$ (decreasing slightly faster than $\sim 1/r^2$) and phase $\varphi_I = \varphi_z = \varphi_p - \varphi_u = \tan^{-1}(1/kr) \sim \tan^{-1}(1) \sim 45^{\mathrm{o}}$.

(c) **In the "far" (or radiation) zone:** $kr \gg 1$, the ***time-averaged*** the 1-D radial complex acoustic intensity is largely ***real*** (i.e. ***active***): $\langle \tilde{I}_{\mathrm{r}}(r) \rangle_t = \left|\tilde{B}\right|^2 / 2z_o r^2$, with magnitude $\langle |\tilde{I}_{\mathrm{r}}(r)| \rangle_t = \left|\tilde{B}\right|^2 / 2z_o r^2$ (decreasing as $\sim 1/r^2$) and phase $\varphi_I = \varphi_z = \Delta\varphi_{p-u} = \varphi_p - \varphi_u = \tan^{-1}(1/kr) \sim \tan^{-1}(0) \sim 0^{\mathrm{o}}$.

The ***time-averaged*** complex acoustic power and its magnitude associated with the point monopole sound source is:

$$\langle \tilde{P}(r) \rangle_t = \int_S \langle \tilde{I}_{\mathrm{r}}(r) \rangle_t \hat{r} \bullet d\vec{S} = 2\pi \frac{\left|\tilde{B}\right|^2}{z_o}\left[1 + \frac{i}{kr}\right] \text{ and: } \langle |\tilde{P}(r)| \rangle_t = 2\pi \frac{\left|\tilde{B}\right|^2}{z_o}\sqrt{1 + \left(\frac{1}{kr}\right)^2}$$

Notice that both the ***time-averaged*** complex acoustic power and its magnitude have an explicit *r*-dependence associated with them – they are not supposed to, after integrating the ***time-averaged*** complex acoustic intensity over a fictitious/Gaussian surface *S* of area $A_{sphere}^{Gauss} = 4\pi r^2$!

The real and imaginary parts of the ***time-averaged*** complex acoustic power are:

$$\langle P_r(r) \rangle_t = \mathrm{Re}\left\{\langle \tilde{P}(r) \rangle_t\right\} = 2\pi \frac{\left|\tilde{B}\right|^2}{z_o} = constant \neq fcn(r)$$

and:

$$\langle P_i(r) \rangle_t = \mathrm{Im}\left\{\langle \tilde{P}(r) \rangle_t\right\} = 2\pi \frac{\left|\tilde{B}\right|^2}{z_o}\frac{1}{kr} \Leftarrow explicit\ fcn(r)!$$

The phase associated with the ***time-averaged*** complex acoustic power is:

$$\varphi_P = \varphi_I = \varphi_z = \Delta\varphi_{p-u} = \varphi_p - \varphi_u = \tan^{-1}(1/kr)$$

In our previous discussions of acoustic power *P* associated e.g. with "point" sound sources, the context was always with regard to ***propagating*** sound – i.e. sound ***radiation*** – purely ***real*** (and ***time-averaged***) acoustic power.

We see from above that the ***real*** component of the ***time-averaged*** complex acoustic power of a monopole sound source – which is associated with ***propagating*** sound energy – is indeed a constant, whereas the ***imaginary*** component of the ***time-averaged*** complex acoustic power of a monopole sound source – which is associated with ***non-propagating*** acoustic energy – ***is*** explicitly *r*-dependent, and especially so in the "near" zone, when $kr \ll 1$.

The purely ***real***, *scalar* ***time-averaged*** potential, kinetic and total energy densities associated with a "point" monopole sound source radiating monochromatic, radially-outgoing spherical waves are:

$$\left\langle w_{potl}(r)\right\rangle_t \equiv \frac{1}{4}\frac{\left|\tilde{p}(r,t)\right|^2}{\rho_o c^2} = \frac{1}{4}\frac{1}{\rho_o c^2}\frac{\left|\tilde{B}\right|^2}{r^2} = \frac{\rho_o\left|\tilde{B}\right|^2}{4z_o^2 r^2} \quad \left(Joules/m^3\right)$$

$$\left\langle w_{kin}(r)\right\rangle_t \equiv \frac{1}{4}\rho_o\left(\tilde{u}_{\rm r}(r,t)\cdot\tilde{u}_{\rm r}^*(r,t)\right) = \frac{1}{4}\rho_o\left|\tilde{u}_{\rm r}(r,t)\right|^2 = \frac{\rho_o\left|\tilde{B}\right|^2}{4z_o^2 r^2}\left[1+\left(\frac{1}{kr}\right)^2\right] \quad \left(Joules/m^3\right)$$

$$\left\langle w_{tot}(r)\right\rangle_t \equiv \left\langle w_{potl}(r)\right\rangle_t + \left\langle w_{kin}(r)\right\rangle_t = \frac{\rho_o\left|\tilde{B}\right|^2}{4z_o^2 r^2} + \frac{\rho_o\left|\tilde{B}\right|^2}{4z_o^2 r^2}\left[1+\left(\frac{1}{kr}\right)^2\right] = \frac{\rho_o\left|\tilde{B}\right|^2}{4z_o^2 r^2}\left[2+\left(\frac{1}{kr}\right)^2\right] \quad \left(Joules/m^3\right)$$

Note that the ratio of ***time-averaged*** potential energy density to ***time-averaged*** kinetic energy density is:

$$\frac{\left\langle w_{potl}(r)\right\rangle_t}{\left\langle w_{kin}(r)\right\rangle_t} = \frac{\dfrac{1}{4}\dfrac{\left|\tilde{p}(r,t)\right|^2}{\rho_o c^2}}{\dfrac{1}{4}\rho_o\left|\tilde{u}_{\rm r}(r,t)\right|^2} = \frac{\dfrac{\cancel{\rho_o}\left|\tilde{B}\right|^2}{4z_o^2\cancel{r^2}}}{\dfrac{\cancel{\rho_o}\left|\tilde{B}\right|^2}{4z_o^2\cancel{r^2}}\left[1+\left(\dfrac{1}{kr}\right)^2\right]} = \frac{1}{\left[1+\left(\dfrac{1}{kr}\right)^2\right]} \le 1$$

In the "near" zone, $kr \ll 1$ where the complex 1-D radial particle velocity, *specific* acoustic impedance and acoustic intensity and power are largely ***reactive*** (i.e. ***imaginary***), $\left\langle w_{potl}(r)\right\rangle_t \ll \left\langle w_{kin}(r)\right\rangle_t$. Only in the "far" (i.e. radiation) zone, when $kr \gg 1$, and moreover, when $kr \to \infty$ (i.e. in free-field conditions) does $\left\langle w_{potl}(r)\right\rangle_t \approx \left\langle w_{kin}(r)\right\rangle_t$.

In terms of the complex relationship between ***time-averaged*** 1-D radial acoustic intensity and ***time-averaged*** total energy density:

$$\left\langle \tilde{I}_{a\,\rm r}(r)\right\rangle_t \equiv \left\langle \tilde{c}_{\rm r}(r)\right\rangle_t \cdot \left\langle w_{tot}(r)\right\rangle_t$$

For the case at hand – the monopole sound source, and again using $z_o = \rho_o c$:

$$\frac{\cancel{\left|\tilde{B}\right|^2}}{\cancel{2}\,\cancel{z_o}\,\cancel{r^2}}\left[1+\frac{i}{kr}\right] = \left\langle \tilde{c}_{\rm r}(r)\right\rangle_t \cdot \frac{\rho_o\cancel{\left|\tilde{B}\right|^2}}{{}_2\cancel{4}\,z_o^{\cancel{2}}\,\cancel{r^2}}\left[2+\left(\frac{1}{kr}\right)^2\right] \quad \text{or:} \quad \left\langle \tilde{c}_{\rm r}(r)\right\rangle_t = c\frac{\left[1+\dfrac{i}{kr}\right]}{\left[1+\dfrac{1}{2}\left(\dfrac{1}{kr}\right)^2\right]}$$

with:

$$\left\langle c_{{\rm r}\,r}(r)\right\rangle_t = \mathrm{Re}\left\{\left\langle \tilde{c}_{\rm r}(r)\right\rangle_t\right\} = c\frac{1}{\left[1+\dfrac{1}{2}\left(\dfrac{1}{kr}\right)^2\right]} \quad \text{and:} \quad \left\langle c_{{\rm r}\,i}(r)\right\rangle_t = \mathrm{Im}\left\{\left\langle \tilde{c}_{\rm r}(r)\right\rangle_t\right\} = c\frac{1}{kr}\frac{1}{\left[1+\dfrac{1}{2}\left(\dfrac{1}{kr}\right)^2\right]}$$

and: $\left\langle \left|\tilde{c}_{\mathrm{r}}(r)\right| \right\rangle_t = c\dfrac{\sqrt{1+\left(\dfrac{1}{kr}\right)^2}}{\left[1+\dfrac{1}{2}\left(\dfrac{1}{kr}\right)^2\right]}$ and: $\varphi_c = \varphi_P = \varphi_I = \varphi_z = \Delta\varphi_{p-u} = \varphi_p - \varphi_u = \tan^{-1}(1/kr)$

In the "near" zone, $kr \ll 1$ where the complex 1-D radial particle velocity, *specific* acoustic impedance and ***time-averaged*** acoustic intensity and power are largely ***reactive*** (i.e. ***imaginary***), we see that the ***time-averaged*** complex 1-D radial c_{r} is also largely ***reactive***:

$$\left\langle \tilde{c}_{\mathrm{r}}(r)\right\rangle_t = c\frac{\left[1+\dfrac{i}{kr}\right]}{\left[1+\dfrac{1}{2}\left(\dfrac{1}{kr}\right)^2\right]} \sim 2i\cdot c\cdot kr \text{ with magnitude } \left\langle \left|\tilde{c}_{\mathrm{r}}(r)\right| \right\rangle_t = c\frac{\sqrt{1+\left(\dfrac{1}{kr}\right)^2}}{\left[1+\dfrac{1}{2}\left(\dfrac{1}{kr}\right)^2\right]} \sim 2c\cdot kr \ll c$$

In the "far" zone, $kr \gg 1$ where the complex 1-D radial particle velocity, *specific* acoustic impedance and ***time-averaged*** acoustic intensity and power are largely ***active*** (i.e. ***real***), we see that the ***time-averaged*** complex 1-D radial c_{r} is also largely ***active***:

$$\left\langle \tilde{c}_{\mathrm{r}}(r)\right\rangle_t = c\frac{\left[1+\dfrac{i}{kr}\right]}{\left[1+\dfrac{1}{2}\left(\dfrac{1}{kr}\right)^2\right]} \sim c \text{ with magnitude } \left\langle \left|\tilde{c}_{\mathrm{r}}(r)\right| \right\rangle_t = c\frac{\sqrt{1+\left(\dfrac{1}{kr}\right)^2}}{\left[1+\dfrac{1}{2}\left(\dfrac{1}{kr}\right)^2\right]} \sim c$$

The corresponding fractional energy densities associated with the ***real*** (i.e. ***propagating***) and ***imaginary*** (i.e. ***non-propagating***) components of the ***time-averaged*** complex 1-D radial acoustic intensity are:

$$\left\langle f_{\mathrm{r}r}(r)\right\rangle_t \equiv \frac{\left\langle c_{\mathrm{r}r}^2(r)\right\rangle_t}{\left\langle \left|\tilde{c}_{\mathrm{r}}(r)\right|^2\right\rangle_t} \text{ and: } \left\langle f_{\mathrm{r}i}(r)\right\rangle_t \equiv \frac{\left\langle c_{\mathrm{r}i}^2(r)\right\rangle_t}{\left\langle \left|\tilde{c}_{\mathrm{r}}(r)\right|^2\right\rangle_t}$$

with:

$$\left\langle c_{\mathrm{r}r}(r)\right\rangle_t = \mathrm{Re}\left\{\left\langle \tilde{c}_{\mathrm{r}}(r)\right\rangle_t\right\} = c\frac{1}{\left[1+\dfrac{1}{2}\left(\dfrac{1}{kr}\right)^2\right]} \text{ and: } \left\langle c_{\mathrm{r}i}(r)\right\rangle_t = \mathrm{Im}\left\{\left\langle \tilde{c}_{\mathrm{r}}(r)\right\rangle_t\right\} = c\frac{1}{kr}\frac{1}{\left[1+\dfrac{1}{2}\left(\dfrac{1}{kr}\right)^2\right]}$$

Thus:

$$\left\langle f_{\mathrm{r}r}(r)\right\rangle_t \equiv \frac{\left\langle c_{\mathrm{r}r}^2(r)\right\rangle_t}{\left\langle \left|\tilde{c}_{\mathrm{r}}(r)\right|^2\right\rangle_t} = \frac{1}{1+\left(\dfrac{1}{kr}\right)^2} \text{ and: } \left\langle f_{\mathrm{r}i}(r)\right\rangle_t \equiv \frac{\left\langle c_{\mathrm{r}i}^2(r)\right\rangle_t}{\left\langle \left|\tilde{c}_{\mathrm{r}}(r)\right|^2\right\rangle_t} = \frac{\left(\dfrac{1}{kr}\right)^2}{1+\left(\dfrac{1}{kr}\right)^2}$$

and thus:

$$\left\langle f_{\mathrm{r}\,r}(r)\right\rangle_t+\left\langle f_{\mathrm{r}\,i}(r)\right\rangle_t=\frac{\left\langle\left|c_r(r)\right|^2\right\rangle_t+\left\langle\left|c_i(r)\right|^2\right\rangle_t}{\left\langle\left|\tilde{c}(r)\right|^2\right\rangle_t}=\frac{1+\left(\frac{1}{kr}\right)^2}{1+\left(\frac{1}{kr}\right)^2}=1 \quad \surd$$

In the "near" zone: $kr \ll 1$, the complex 1-D radial particle velocity, *specific* acoustic impedance, ***time-averaged*** acoustic intensity and power, and ***time-averaged*** complex 1-D radial c_r are all largely ***reactive*** (i.e. ***imaginary***). Using appropriate Taylor's series expansions, we see that:

$$\left\langle f_{\mathrm{r}\,r}(r)\right\rangle_t=\frac{1}{1+\left(\frac{1}{kr}\right)^2}\sim(kr)^2\ll 1 \;\text{ and: }\; \left\langle f_{\mathrm{r}\,i}(r)\right\rangle_t=\frac{\left(\frac{1}{kr}\right)^2}{1+\left(\frac{1}{kr}\right)^2}\sim 1-(kr)^2\approx 1$$

In the "far" zone: $kr \gg 1$, the complex 1-D radial particle velocity, *specific* acoustic impedance, ***time-averaged*** acoustic intensity and power, and ***time-averaged*** complex 1-D radial c_r are all largely ***active*** (i.e. ***real***). Using appropriate Taylor's series expansions, we see that:

$$\left\langle f_{\mathrm{r}\,r}(r)\right\rangle_t=\frac{1}{1+\left(\frac{1}{kr}\right)^2}\sim 1-\left(\frac{1}{kr}\right)^2\approx 1 \;\text{ and: }\; \left\langle f_{\mathrm{r}\,i}(r)\right\rangle_t=\frac{\left(\frac{1}{kr}\right)^2}{1+\left(\frac{1}{kr}\right)^2}\sim\left(\frac{1}{kr}\right)^2\ll 1$$

Example # 4: Physical "Point" Monopole Sound Source:

At very low frequencies a loudspeaker in a fully-enclosed/sealed cabinet of characteristic dimension *a*, with $ka \ll 1$ (i.e. $f \ll c/2\pi a$ {using $k\equiv 2\pi/\lambda$, $\omega\equiv 2\pi f$ and $c=f\lambda=\omega/k$ }) approximates a "point" monopole sound source – the directivity factor, *Q* of a typical enclosed loudspeaker is very nearly 1 (i.e. isotropic) at frequencies $f \ll c/2\pi a$. For example, for a typical "bookshelf"-type loudspeaker with *a* ~ 1 *ft* ~ 0.3 *m*, then $f \ll c/2\pi a = 343/0.6\pi \sim 180\ Hz$, or equivalently $\lambda \gg 1.9\ m$.

If we use the "spherical cow" approximation, i.e. model a physical monopole sound source as a radially-pulsating sphere of radius *a*, subject to the restriction $ka \ll 1$, then the acoustical properties of such a device (for *r* > *a*) will closely approximate that of an ideal, point monopole sound source.

How do we characterize the *strength* of a physical monopole sound source – i. e. a radially-pulsating sphere of radius *a*? Typically, this is done by considering the {complex} *volumetric velocity* (*aka* *volume velocity*) of the physical monopole source, evaluated at the radius *a* of the sphere – the {radial} outward volume rate (or flow) of fluid (i.e. air) from this sphere:

$$\tilde{Q}_a e^{i\omega t}=\int_S \tilde{u}_\mathrm{r}(r=a,t)\,\hat{r}\bullet\hat{n}\,dS=4\pi a\frac{\tilde{B}}{z_o}\left[1-\frac{i}{ka}\right]e^{i\omega t}e^{-ika}\ \left(m^3/s\right)$$

Thus the {complex} source strength/volume velocity of a physical monopole is:

$$\tilde{Q}_a = 4\pi a \frac{\tilde{B}}{z_o}\left[1-\frac{i}{ka}\right]e^{-ika}\ \left(m^3/s\right)$$

Since $ka \ll 1$, then $1/ka \gg 1$and we can approximate this expression as:

$$\tilde{Q}_a \simeq -4\pi i\cdot\frac{\tilde{B}}{z_o k}\left\{\underbrace{\cos ka}_{\approx 1} - i\underbrace{\sin ka}_{\approx 0}\right\} \simeq -4\pi i\cdot\frac{\tilde{B}}{z_o k} = -4\pi i\cdot\frac{\tilde{B}}{\rho_o ck} = -4\pi i\cdot\frac{\tilde{B}}{\rho_o\omega}\ \left(m^3/s\right)$$

Thus, we see that: $\tilde{B} \simeq i\dfrac{\rho_o\omega}{4\pi}\tilde{Q}_a\ \left(Pascal\text{-}m\right)$

Expressed in terms of the complex source strength/volume velocity $\tilde{Q}_a$ of the physical monopole sphere of radius *a*, the complex pressure and 1-D radial particle velocity associated with this sound source are (for *r* > *a*):

$$\tilde{p}(r,t) = \frac{\tilde{B}}{r}e^{i(\omega t-kr)} = i\frac{\rho_o\omega}{4\pi}\frac{\tilde{Q}_a}{r}e^{i(\omega t-kr)}$$

$$\tilde{u}_{\mathrm{r}}(r,t) = \frac{1}{z_o}\frac{\tilde{B}}{r}\left[1-\frac{i}{kr}\right]e^{i(\omega t-kr)} = i\frac{\omega}{4\pi c}\frac{\tilde{Q}_a}{r}\left[1-\frac{i}{kr}\right]e^{i(\omega t-kr)}$$

The complex 1-D radial <u>*specific*</u> acoustic impedance is (unchanged):

$$\tilde{z}_{\mathrm{r}}(r) \equiv \frac{\tilde{p}(r,t)}{\tilde{u}_{\mathrm{r}}(r,t)} = z_o\frac{1}{\left[1-i/kr\right]} = z_o\frac{\left[1+i/kr\right]}{\left[1+\left(1/kr\right)^2\right]}$$

The <u>***time-averaged***</u> complex 1-D radial acoustic intensity (for *r* > *a*) is:

$$\left\langle\tilde{I}_{\mathrm{r}}(r)\right\rangle_t \equiv \frac{1}{2}\left\langle\tilde{p}(r,t)\tilde{u}_{\mathrm{r}}^*(r,t)\right\rangle_t = \frac{1}{2}\frac{1}{z_o}\frac{\left|\tilde{B}\right|^2}{r^2}\left[1+\frac{i}{kr}\right] = \frac{\rho_o\omega^2}{32\pi^2 c}\frac{\left|\tilde{Q}_a\right|^2}{r^2}\left[1+\frac{i}{kr}\right]$$

Note that $\left\langle\tilde{I}_{\mathrm{r}}(r)\right\rangle_t \propto f^2$ - i.e. a monopole sound source is <u>***not***</u> efficient at very low frequencies in terms of generating sound….

The <u>***time-averaged***</u> complex acoustic power associated with this monopole sound source (for *r* > *a*) is:

$$\left\langle\tilde{P}(r)\right\rangle_t = \int_S\left\langle\tilde{I}_{\mathrm{r}}(r)\right\rangle_t \hat{r}\bullet d\vec{S} = 2\pi\frac{\left|\tilde{B}\right|^2}{z_o}\left[1+\frac{i}{kr}\right] = \rho_o\omega^2\frac{\left|\tilde{Q}_a\right|^2}{8\pi c}\left[1+\frac{i}{kr}\right]$$

The ***time-averaged*** potential, kinetic and total energy densities associated with the physical monopole sound source (for $r > a$) are:

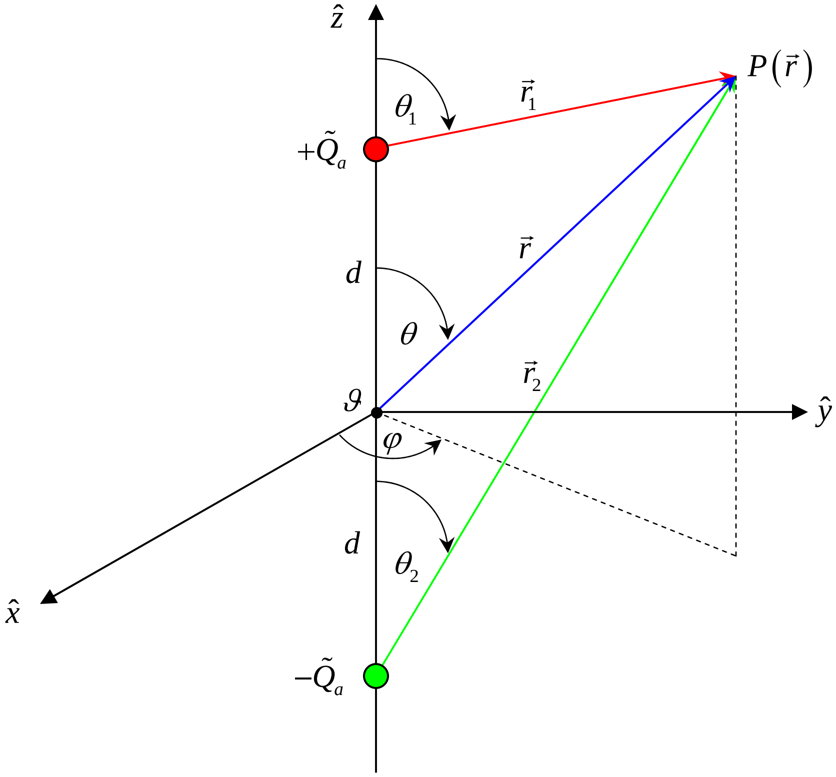
$$\langle w_{potl}(r)\rangle_t \equiv \frac{1}{4}\frac{|\tilde{p}(r,t)|^2}{\rho_o c^2} = \frac{\rho_o\omega^2}{64\pi^2c^2}\frac{|\tilde{Q}_a|^2}{r^2} \quad (Joules/m^3)$$

$$\langle w_{kin}(r)\rangle_t \equiv \frac{1}{4}\rho_o\left(\tilde{u}_{\mathrm{r}}(r,t)\cdot\tilde{u}_{\mathrm{r}}^*(r,t)\right) = \frac{1}{4}\rho_o|\tilde{u}_{\mathrm{r}}(r,t)|^2 = \frac{\rho_o\omega^2}{64\pi^2c^2}\frac{|\tilde{Q}_a|^2}{r^2}\left[1+\left(\frac{1}{kr}\right)^2\right] \quad (Joules/m^3)$$

$$\langle w_{tot}(r)\rangle_t \equiv \langle w_{potl}(r)\rangle_t + \langle w_{kin}(r)\rangle_t = \frac{\rho_o\omega^2}{64\pi^2c^2}\frac{|\tilde{Q}_a|^2}{r^2}\left[2+\left(\frac{1}{kr}\right)^2\right] \quad (Joules/m^3)$$

Example # 5: Compact/Physical Dipole Sound Source:

By the principle of linear superposition, we can create a so-called compact/physical dipole sound source using two ***out-of-phase*** physical monopole sources, of source strength/volume velocity $\pm\tilde{Q}_a$, and separated from each other by a distance 2*d*, and subject to the requirement that $kd \ll 1$ (i.e. $f \ll c/2\pi d$ or $d \ll c/2\pi f$), as shown in the figure below:



The total/resultant complex over-pressure at the observation/listener's point $P(\vec{r})$ in the above figure is the linear sum of the individual complex over-pressures associated with each monopole source:

$$\tilde{p}_{tot}(\vec{r},t) = \tilde{p}_1(\vec{r},t) + \tilde{p}_2(\vec{r},t) = \tilde{B}\left[\frac{1}{r_1}e^{-ikr_1} - \frac{1}{r_2}e^{-ikr_2}\right]e^{i\omega t} = i\frac{\rho_o\omega\tilde{Q}_a}{4\pi}\left[\frac{1}{r_1}e^{-ikr_1} - \frac{1}{r_2}e^{-ikr_2}\right]e^{i\omega t}$$

The total/resultant complex particle velocity at the observation/listener's point $P(\vec{r})$ in the above figure is the ***vector*** sum of the individual complex particle velocities associated with each monopole source:

$$\begin{aligned}\vec{\tilde{u}}_{tot}(\vec{r},t) &= \vec{\tilde{u}}_1(\vec{r},t) + \vec{\tilde{u}}_2(\vec{r},t) = \tilde{u}_1(\vec{r},t)\hat{r}_1 + \tilde{u}_2(\vec{r},t)\hat{r}_2 \\ &= \quad \frac{\tilde{B}}{z_o}\left\{\frac{1}{r_1}\left[1-\frac{i}{kr_1}\right]e^{-ikr_1}\hat{r}_1 - \frac{1}{r_2}\left[1-\frac{i}{kr_2}\right]e^{-ikr_2}\hat{r}_2\right\}e^{i\omega t} \quad using{:}\ \ \tilde{B} \simeq i\frac{\rho_o\omega}{4\pi}\tilde{Q}_a \\ &= i\frac{\omega\tilde{Q}_a}{4\pi c}\left\{\frac{1}{r_1}\left[1-\frac{i}{kr_1}\right]e^{-ikr_1}\hat{r}_1 - \frac{1}{r_2}\left[1-\frac{i}{kr_2}\right]e^{-ikr_2}\hat{r}_2\right\}e^{i\omega t} \quad and{:} \quad z_o = \rho_o c\end{aligned}$$

The acoustic monopole sources, of source strength/volume velocity $\pm\tilde{Q}_a$ are located at $\vec{d}_1 = +d\hat{z}$ and $\vec{d}_2 = -d\hat{z}$ with $|\vec{d}_1| = |\vec{d}_2| = d$. Vectorially, we see that $\vec{r} = \vec{d}_1 + \vec{r}_1$ and also that $\vec{r} = \vec{d}_2 + \vec{r}_2$, with $\vec{r} = r\hat{r}$ and $|\vec{r}| = r = \sqrt{x^2+y^2+z^2}$. In Cartesian coordinates $\hat{r} = \sin\theta\cos\varphi\hat{x} + \sin\theta\sin\varphi\hat{y} + \cos\theta\hat{z}$. We also see that: $\vec{r}_1 = \vec{r} - \vec{d}_1$ and $\vec{r} = \vec{r}_2 - \vec{d}_2$. Using the law of cosines $c^2 = a^2 + b^2 - 2ab\cos\theta$: $|\vec{r}_1| = r_1 = \sqrt{r^2 + d^2 - 2rd\cos\theta}$ and $|\vec{r}_2| = r_2 = \sqrt{r^2 + d^2 - 2rd\cos(\pi-\theta)} = \sqrt{r^2 + d^2 + 2rd\cos\theta}$, with $\vec{r}_1 = |\vec{r}_1|\hat{r}_1 = r_1\hat{r}_1 = \vec{r} - \vec{d}_1 = r\hat{r} - d\hat{z}$ and $\vec{r}_2 = |\vec{r}_2|\hat{r}_2 = r_2\hat{r}_2 = \vec{r} - \vec{d}_2 = r\hat{r} + d\hat{z}$ with $\hat{r}_1 = (r\hat{r} - d\hat{z})/r_1$ and $\hat{r}_2 = (r\hat{r} + d\hat{z})/r_2$.

Since $\hat{r}_1 \neq \hat{r}_2 \neq \hat{r}$ (especially in the "near"-field region), the above expression for the total / resultant complex ***vector*** particle velocity $\vec{\tilde{u}}_{tot}(\vec{r},t)$ is not easy to evaluate, analytically. However, note that it <u>does</u> simplify <u>considerably</u> in the "far"-field region, where $\hat{r}_1 \simeq \hat{r}_2 \simeq \hat{r}$ and $r_1 \simeq r_2 \simeq r \gg d$.

It is quite clear from the above formulae for $\tilde{p}_{tot}(\vec{r},t)$ and $\vec{\tilde{u}}_{tot}(\vec{r},t)$ {as well as from previous P498POM lectures on sound interference effects with 2 (or more) sources} that interference effects will indeed manifest themselves here in this situation, albiet in a much more complicated manner….

However, the nature of this problem is such that *<u>all</u>* of the above quantities that we have calculated analytically e.g. for the various simpler sound sources can be also easily coded up on a computer, e.g. using MATLAB, Mathematica or e.g. a C/C++ based-program coupled to a graphics software package for plots, not just for them, but for this problem as well…

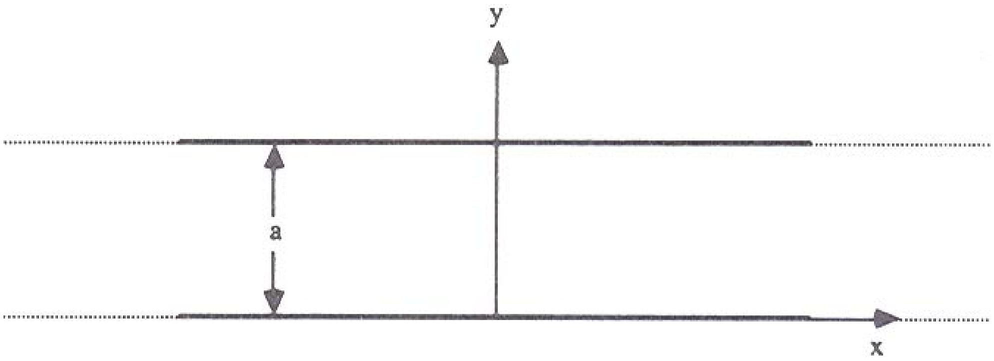
Computational calculations can also be done for {arbitrarily} higher-order acoustic multipoles – e.g. linear {and/or crossed (*aka* lateral)} quadrupoles (the tuning fork is an example of a linear quadrupole), sextupoles, octupoles, hexadecapoles, arbitrary linear 1-D acoustic arrays, 2-D/3-D acoustic arrays, all using the principle of linear superposition for *N* monopole sound sources…

Some very nice animation demos of pressure fields for monopole, dipole, quadrupole… sound sources exist e.g. at Prof. Dan Russell's website:

http://www.kettering.edu/~drussell/Demos/rad2/mdq.html

Example # 6: The Uniform Planar Rigid-Walled 2-D Duct:

The final example we wish to discuss is that of sound propagation of monochromatic waves in an infinitely-long uniform planar duct consisting of two infinite, parallel and rigid walls separated by a perpendicular distance *a*, as shown in the figure below:



The wave equation for the complex scalar pressure field is:

$$\nabla^2 \tilde{p}(\vec{r},t) - \frac{1}{c^2}\frac{\partial^2 \tilde{p}(\vec{r},t)}{\partial t^2} = 0$$

The sound propagation direction is in the $+\hat{x}$ -direction; note that the sound waves are not constrained in the *z*-direction ($+\hat{z}$ points out of the page), whereas sound waves *are* constrained in the *y*-direction, being allowed only in the region between the two infinite, parallel walls: $0 \le y \le a$. Thus, this problem is only a 2-D problem in (x, y) rectangular coordinates.

The gradient $\vec{\nabla}$ and Laplacian ∇^2 operators in 2-D Cartesian/rectangular coordinates are:

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} \quad \text{and:} \quad \nabla^2 \equiv \vec{\nabla}\bullet\vec{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

The 2-D wave equation for the complex scalar pressure field is thus:

$$\frac{\partial^2 \tilde{p}(x,y,t)}{\partial x^2} + \frac{\partial^2 \tilde{p}(x,y,t)}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2 \tilde{p}(x,y,t)}{\partial t^2} = 0$$

We seek $+\hat{x}$-propagating wave product-type solutions of the general form:

$$\tilde{p}(x,y,t) = \tilde{X}(x)\tilde{Y}(y)\tilde{T}(t) = Ae^{-ik_x x}e^{\mp ik_y y}e^{i\omega t}$$

The homogeneous wave equation is separable in these variables; the resulting *characteristic equation* for the wavenumber *k* is: $k^2 = k_x^2 + k_y^2$, with accompanying *dispersion relation* $\omega^2 = c^2 k^2$. {The details of this separation-of-variables technique for the 2-D wave equation are given in the P498POM Lecture Notes on "Mathematical Musical Physics of the Wave Equation" p. 12-13}.

The boundary condition on the pressure at the two infinite, rigid parallel walls in the $\hat{y}$-direction is that there are pressure *anti-nodes* at *y* = 0 and *y* = *a*. Mathematically, this requires Neumann-type boundary conditions on the walls, i.e. $\partial\tilde{p}(x,y=0,t)/\partial y = \partial\tilde{p}(x,y=a,t)/\partial y = 0$, requiring cosine-type solutions for $\tilde{Y}(y) = e^{\mp ik_y y}$, i.e. $\tilde{Y}(y) \sim e^{ik_y y} + e^{-ik_y y} \sim \cos k_y y$ such that:

$$\cos k_y y\Big|_{y=0,a} = 1 \text{ with } \partial\cos k_y y/\partial y\Big|_{y=0,a} = \sin k_y y\Big|_{y=0,a} = 0 \;\Rightarrow\; k_y a = n\pi,\; n = 0,1,2,3...$$

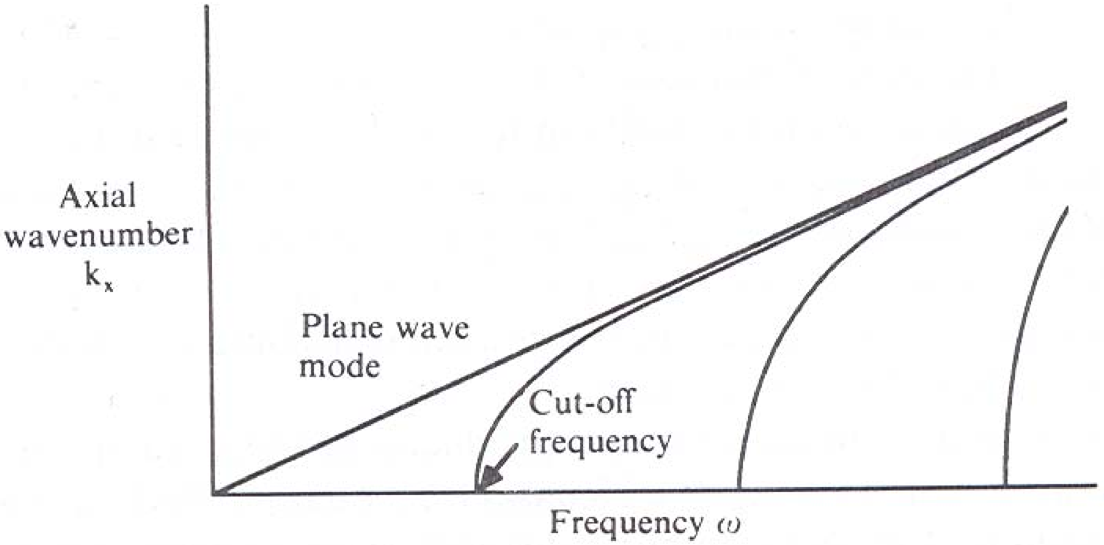
Thus we see that $k_y a = n\pi$ or $k_y = n\pi/a$, $n = 0,1,2,3....$ Thus, for *any* given frequency $f = \omega/2\pi$, there are an *infinite* number of possible solutions (*aka* eigenmodes) for this wave equation, each one of the general form:

$$\tilde{p}_n(x,y,t) = \tilde{A}_n\cos(n\pi y/a)e^{i(\omega t - k_n x)} \text{ where: } k_x = \sqrt{k^2 - k_y^2} \;\Rightarrow\; k_n = \sqrt{k^2 - (n\pi/a)^2}$$

The transverse pressure distribution $\sim\cos(n\pi y/a)$ for $0 \le y \le a$ is a characteristic of the wall geometry associated with this problem – i.e. a ***duct***; and one which is caused by multiple, perfect (i.e lossless) reflections of the pressure waves off of the duct walls as they propagate in the $+\hat{x}$-direction. The integer *n* denotes the {duct-} mode of propagation. The *n* = 0 mode is known as the ***axial*** plane-wave eigenmode of propagation. The $n \ge 1$ modes are collectively known as ***transverse*** duct eigenmodes. At a given frequency *f*, if a specific duct eigenmode *n* is excited, it may only propagate along the duct with a unique *axial* wavenumber given by $k_n = \sqrt{k^2 - (n\pi/a)^2}$.

Note that for each/every duct eigenmode *n* of propagation, there is an (angular) frequency ω at which the axial wavenumber $k_n = \sqrt{k^2 - (n\pi/a)^2} = \sqrt{(\omega/c)^2 - (n\pi/a)^2} = 0$. The so-called cutoff frequency for the *n*th mode is: $\omega_n^{cutoff} = n\pi c/a$ or: $f_n^{cutoff} = nc/2a$. Below this cutoff frequency, the duct eigenmode n cannot propagate – it becomes an *evanescent* mode because the axial eigen-wavenumber k_n becomes purely imaginary for $f < f_n^{cutoff} = nc/2a$ – i.e. the duct eigenmode *n* is exponentially attenuated by a factor of $e^{-k_n x}$ when $f < f_n^{cutoff} = nc/2a$.

A plot of the *dispersion curves* - axial wavenumber $k_x = k_n$ *vs*. angular frequency ω showing the effect of the cutoff frequency *vs*. mode number *n* = 0, 1, 2, 3, … is shown in the figure below.



For a given (angular) frequency ω, the *total/net* complex pressure wave is a sum over *all* modes – the allowed/*propagating* individual complex pressure eigenmodes $n \le n_{cutoff}$, where n_{cutoff} is the *highest* eigenmode number *n* such that $k_x = k_n = \sqrt{(\omega/c)^2 - (n_{cutoff}\pi/a)^2} > 0$, i.e. $n_{cutoff} = \text{int}\{\omega a/\pi c\}$ $(= \text{floor}\{\omega a/\pi c\})$, as well as the individual *non-propagating* modes $n > n_{cutoff}$:

$$\tilde{p}_{tot}(x,y,t) = \sum_{n=0}^{\infty} \tilde{p}_n(x,y,t) = \sum_{n=0}^{\infty} \tilde{A}_n \cos(n\pi y/a) e^{i(\omega t - k_n x)}$$

{n.b. Far from the source, this reduces to the sum over *propagating* modes $n \le n_{cutoff}$.}

The complex 2-D particle velocity $\vec{\tilde{u}}(\vec{r},t)$ associated with this problem is {again} determined via use of Euler's equation. Since $\tilde{p}(\vec{r},t) = \tilde{p}(x,y,t) \ne fcn(z)$, then $\vec{\nabla}\tilde{p}(\vec{r},t) = \vec{\nabla}\tilde{p}(x,y,t) = (\partial\tilde{p}(x,y,t)/\partial x)\hat{x} + (\partial\tilde{p}(x,y,t)/\partial y)\hat{y}$ and thus the particle velocity can only be in the (x,y) direction(s), i.e. $\vec{u}(\vec{r},t) = \vec{u}(x,y,t) \ne fcn(z)$ since:

$$\frac{\partial\vec{u}(\vec{r},t)}{\partial t} = -\frac{1}{\rho_o}\vec{\nabla}p(\vec{r},t) \;\Rightarrow\; \frac{\partial\vec{\tilde{u}}(x,y,t)}{\partial t} = -\frac{1}{\rho_o}\left(\frac{\partial\tilde{p}(x,y,t)}{\partial x}\hat{x} + \frac{\partial\tilde{p}(x,y,t)}{\partial y}\hat{y}\right)$$

Euler's equation holds for each/every duct eignmode *n*. With $\tilde{p}_n(x,y,t) = \tilde{A}_n \cos(n\pi y/a) e^{i(\omega t - k_n x)}$, the general form of the 2-D complex particle velocity for the n^{th} duct eigenmode is thus:

$$\vec{\tilde{u}}_n(x,y,t) = \frac{1}{\omega\rho_o}\tilde{A}_n\left[k_n \cos(n\pi y/a)\hat{x} - i(n\pi/a)\sin(n\pi y/a)\hat{y}\right]e^{i(\omega t - k_n x)}$$

$$\text{where: } k_x = k_n = \sqrt{k^2 - (n\pi/a)^2} = \sqrt{(\omega/c)^2 - (n\pi/a)^2}$$

For a given (angular) frequency ω, for each of the propagating duct eigenmodes $n \le n_{cutoff}$, the ***axial*** component of the particle velocity $\tilde{u}_{n_x}(x,y,t)\,\hat{x} = \frac{1}{\omega\rho_o}\tilde{A}_n k_n \cos(n\pi y/a)e^{i(\omega t - k_n x)}\hat{x}$ is ***in-phase*** with the complex pressure $\tilde{p}_n(x,y,t) = \tilde{A}_n \cos(n\pi y/a)e^{i(\omega t - k_n x)}$, whereas the complex ***transverse*** component of the particle velocity $\tilde{u}_{n_y}(x,y,t)\,\hat{y} = \frac{-i}{\omega\rho_o}\tilde{A}_n (n\pi/a)\sin(n\pi y/a)e^{i(\omega t - k_n x)}\hat{y}$ is in ***quadrature*** (i.e. 90^{o} out of phase) with the complex pressure.

The *total/net* 2-D particle velocity is likewise given by:

$$\vec{\tilde{u}}(x,y,t) = \sum_{n=0}^{\infty}\vec{\tilde{u}}_n(x,y,t) = \frac{1}{\omega\rho_o}\sum_{n=0}^{\infty}\tilde{A}_n\left[k_n\cos(n\pi y/a)\,\hat{x} - i(n\pi/a)\sin(n\pi y/a)\,\hat{y}\right]e^{i(\omega t - k_n x)}$$

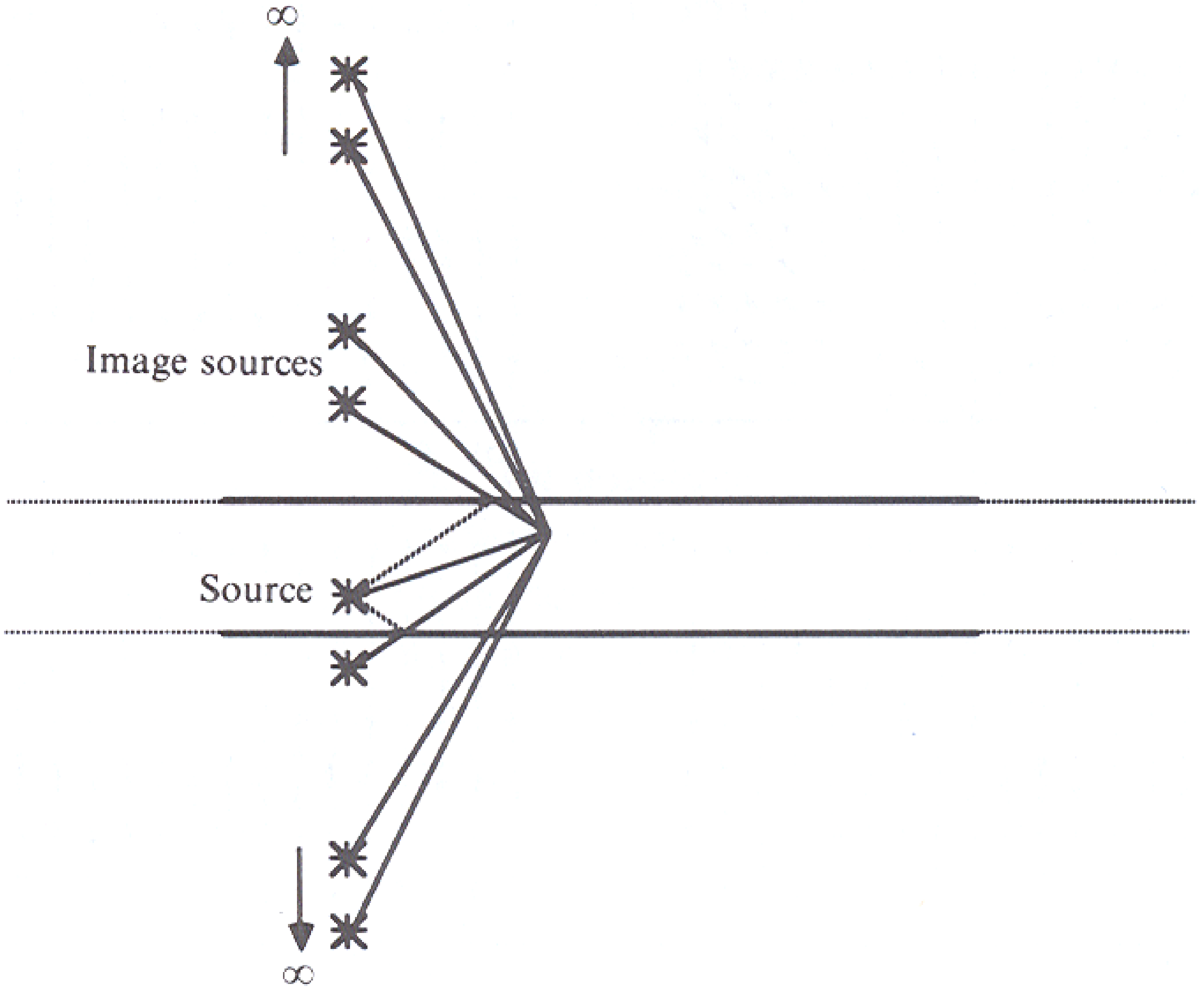
where: $k_x = k_n = \sqrt{k^2 - (n\pi/a)^2} = \sqrt{(\omega/c)^2 - (n\pi/a)^2}$ and: $n_{cutoff} = \text{int}\{\omega a/\pi c\}$ $(= \text{floor}\{\omega a/\pi c\})$.

The total/net complex pressure wave $\tilde{p}_{tot}(x,y,t) = \sum_{n=0}^{\infty}\tilde{p}_n(x,y,t)$ and 2-D particle velocity wave $\vec{\tilde{u}}(x,y,t) = \sum_{n=0}^{\infty}\vec{\tilde{u}}_n(x,y,t)$ that propagate in a duct depend on the details of the coupling of the sound *source* to that duct. For example, a 2-D "line" monopole source of volumetric velocity *per unit length* $Q_a' = Q_a/L$ (m^2/s) located e.g. at $(x,y) = (0,y_o)$ in the duct will produce modal pressure amplitudes of:

$$\tilde{A}_n = \frac{\omega\rho_o Q_a'}{k_n a}\cos(n\pi y_o/a)$$

Note that this relation predicts that the n^{th} modal pressure amplitude $\tilde{A}_n$ becomes ***infinite*** at the cutoff frequency for that mode, $\omega_n^{cutoff} = n\pi c/a$ when $k_x = k_n = \sqrt{(\omega_{cutoff}/c)^2 - (n\pi/a)^2} = 0$!!! However, in the real world, nothing becomes infinite – e.g. the *finite* impedance of a real acoustic source will preclude infinite acoustic energy transfer to the duct. Nevertheless, real / experimental modal pressure amplitudes $\tilde{A}_n$ do indeed become large at/near the cutoff frequency!

The method of (an infinite set of) acoustic images can be used to model sound sources inside of (perfectly reflecting – even for partially reflecting) ducts – the planar walls of the duct act like mirrors, thus virtual "images" of the sound source in the duct are produced outside of the duct, as shown in the figure below:



The pressure/particle velocity fields close/in proximity to the actual sound source inside the duct are determined largely by the image source(s) nearest to the actual sound source; the solution converges rapidly as the number of image sources is increased. However, accuracy in calculating the pressure / particle velocity fields far from the actual sound source (i.e. further down the duct, $\Delta x \gg a$) requires increasingly larger numbers of image sources to be included. The image source technique is widely used e.g. in computational modeling of room acoustics.

The 2-D vector complex specific impedance/admittance, time-averaged acoustic intensity and scalar energy densities, etc. inside the duct can all be computed (best accomplished e.g. via numerical computation…) from their definitions, for a given sound source & frequency:

$$\vec{\tilde{z}}_a(x,y) \equiv \frac{\tilde{p}(x,y,t)}{\vec{\tilde{u}}(x,y,t)} = \frac{\tilde{p}(x,y,t)\cdot\vec{\tilde{u}}^*(x,y,t)}{\left|\vec{\tilde{u}}(x,y,t)\right|^2} \quad (\Omega_a) \quad \text{and:} \quad \vec{\tilde{y}}(x,y) \equiv \frac{\vec{\tilde{u}}(x,y,t)}{p(x,y,t)} \quad \left(\Omega_a^{-1}\right)$$

$$\left\langle \vec{\tilde{I}}_a(x,y) \right\rangle_t \equiv \frac{1}{2}\tilde{p}(x,y,t)\cdot\vec{\tilde{u}}^*(x,y,t) \quad \left(Watts/m^2\right)$$

$$\left\langle w_{potl}\left(x,y\right)\right\rangle_t \equiv \frac{1}{4}\frac{\left|\tilde{p}\left(x,y,t\right)\right|^2}{\rho_o c^2} \quad \left(Joules/m^3\right)$$

$$\left\langle w_{kin}\left(x,y\right)\right\rangle_t \equiv \frac{1}{4}\rho_o\left(\vec{\tilde{u}}\left(x,y,t\right)\cdot\vec{\tilde{u}}^*\left(x,y,t\right)\right) \quad \left(Joules/m^3\right)$$

$$\left\langle w_{tot}\left(x,y\right)\right\rangle_t \equiv \left\langle w_{potl}\left(x,y\right)\right\rangle_t + \left\langle w_{kin}\left(x,y\right)\right\rangle_t \quad \left(Joules/m^3\right)$$

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