Examples of Complex Sound Fields:

Example # 1: 1-D Plane Monochromatic Traveling Wave Propagating in "Free Air":

Note: this acoustic example is the electrical analog of a simple *AC* circuit with a purely real instantaneous *AC* voltage $V(t) = V_o \cos \omega t$ imposed across a purely real resistor of resistance R (Ω) with purely real instantaneous *AC* current $I(t) = I_o \cos \omega t$ flowing through it.

In "free air", the instantaneous pressure at a space-time point (x,t) associated with a 1-D plane monochromatic traveling wave propagating e.g. in the +*x*-direction is a purely real quantity: $p(x,t) = p_o \cos(\omega t - kx)$.

The 1-D instantaneous longitudinal particle velocity (i.e. in the *x*-/propagation direction) at the space-time point (x,t) associated with a 1-D plane monochromatic traveling wave is obtained via use of the 1-D Euler equation:

$$\frac{\partial u^{\parallel}(x,t)}{\partial t} = -\frac{1}{\rho_o}\frac{\partial p(x,t)}{\partial x} = -\frac{p_o}{\rho_o}\frac{\partial \cos(\omega t - kx)}{\partial x} = -\frac{kp_o}{\rho_o}\sin(\omega t - kx)$$

Then:

$$u^{\parallel}(x,t) = -\frac{kp_o}{\rho_o}\int \sin(\omega t - kx)\,dt = +\frac{kp_o}{\omega\rho_o}\cos(\omega t - kx) = \frac{p_o}{\rho_o c}\cos(\omega t - kx) = u_o^{\parallel}\cos(\omega t - kx)$$

where we have used the relation $c = \omega/k = 343\,m/s$ = speed of sound in {bone-dry} air @ NTP. Note also that $u_o^{\parallel} = p_o/\rho_o c$ /

Since $p(x,t) = p_o \cos(\omega t - kx)$ and $u^{\parallel}(x,t) = (p_o/\rho_o c)\cos(\omega t - kx) = u_o^{\parallel}\cos(\omega t - kx)$, we see that the pressure and 1-D longitudinal particle velocity are *in-phase* with each other for a 1-D monochromatic plane wave propagating in "free" air. This in turn implies that the 1-D longitudinal *specific* acoustic impedance, admittance and sound intensity will thus also be purely *real* quantities for a 1-D monochromatic plane wave propagating in "free" air…

We can express the pressure and 1-D longitudinal particle velocity in complex notation as: $p(x,t) = p_o e^{i(\omega t - kx)}$ and $u^{\parallel}(x,t) = u_o^{\parallel} e^{i(\omega t - kx)}$. The 1-D longitudinal *specific* acoustic impedance associated with a 1-D monochromatic plane wave propagating e.g. in the +*x*-direction in "free" air is then easily seen to be a purely real quantity:

$$\tilde{z}_a^{\parallel}(x) = \frac{\tilde{p}(x,t)}{\tilde{u}^{\parallel}(x,t)} = \frac{p_o \cancel{e^{i(\omega t - kx)}}}{u_o^{\parallel} \cancel{e^{i(\omega t - kx)}}} = \frac{p_o}{u_o^{\parallel}} = \frac{\cancel{p_o}}{\cancel{p_o}/\rho_o c} = \rho_o c \equiv z_o \ \ (\Omega_a)$$

The purely ***real*** quantity $z_o \equiv \rho_o c = 1.204 \cdot 343 \simeq 413 \ (\Omega_a)$ @ NTP is known as the ***characteristic specific acoustic impedance*** of free air. Its inverse is the purely real ***characteristic specific acoustic admittance*** of free air: $y_o = 1/z_o = 1/\rho_o c \simeq 1/413 \simeq 2.42\times10^{-3} \ \left(\Omega_a^{-1}\right)$. Note that neither c, ρ_o, z_o nor y_o are actually constant e.g. with air temperature, T as shown in the table below, for an ambient pressure of P_{atm} = 1 atmosphere:

Temperature (^{o}C)	c (m/s)	ρ_o (kg/m^3)	z_a (Ω_a)	y_a (Ω_a)
–10	325.2	1.342	436.1	2.293×10^{-3}
–5	328.3	1.317	432.0	2.315×10^{-3}
0	331.3	1.292	428.4	2.334×10^{-3}
+5	334.3	1.269	424.3	2.357×10^{-3}
+10	337.3	1.247	420.6	2.378×10^{-3}
+15	340.3	1.225	416.8	2.399×10^{-3}
+20	343.2	1.204	413.2	2.420×10^{-3}
+25	346.1	1.184	409.8	2.440×10^{-3}
+30	349.0	1.165	406.3	2.461×10^{-3}

Note that the 1-D longitudinal ***specific*** acoustic impedance $z_a^{\parallel}(x) = \rho_o c \equiv z_o \ (\Omega_a)$ and/or 1-D longitudinal ***specific*** acoustic admittance $y_o^{\parallel} = 1/z_o^{\parallel} = 1/\rho_o c \ \left(\Omega_a^{-1}\right)$ associated with a 1-D monochromatic plane wave also have ***no*** spatial (i.e. x-) and/or frequency (i.e. f -) dependence.

The ***instantaneous*** 1-D longitudinal sound intensity associated with a 1-D monochromatic plane traveling wave propagating e.g. in the +x-direction in "free" air is also a purely ***real*** quantity. The instantaneous real/in-phase/active and imaginary/quadrature/reactive components respectively, are:

$$I_{a\,r}^{\parallel inst}(x,t) \equiv p_r(x,t)\cdot u_r^{\parallel}(x,t) = p_o u_o^{\parallel} \cos^2(\omega t - kx)$$
$$I_{a\,i}^{\parallel inst}(x,t) \equiv p_r(x,t)\cdot u_i^{\parallel}(x,t) = p_o u_o^{\parallel} \cos(\omega t - kx)\sin(\omega t - kx)$$

For an observer's/listener's position e.g. at x = 0, these becomes:

$$I_{a\,r}^{\parallel inst}(x=0,t) = p(x=0,t)u^{\parallel}(x=0,t) = p_o u_o^{\parallel} \cos^2 \omega t$$
$$I_{a\,i}^{\parallel inst}(x=0,t) = p(x=0,t)u^{\parallel}(x=0,t) = p_o u_o^{\parallel} \cos \omega t \sin \omega t$$

Noting that the time-averaged $\left\langle \cos^2 \omega t \right\rangle_t \equiv \dfrac{1}{\tau}\int_{t=0}^{t=\tau} \cos^2 \omega t \, dt = \tfrac{1}{2}$ and that the time-averaged $\left\langle \cos \omega t \sin \omega t \right\rangle_t \equiv \tfrac{1}{\tau}\int_{t=0}^{t=\tau} \cos \omega t \sin \omega t \, dt = 0$, the ***time-averaged*** real/in-phase/active and imaginary/quadrature/reactive components of the instantaneous 1-D longitudinal sound intensity at the listener's position x = 0 associated with a 1-D monochromatic plane traveling wave propagating e.g. in the +x-direction in "free" air respectively, are:

$$\left\langle I_{a\,r}^{\|\,inst}\left(x=0\right)\right\rangle_t = p_o u_o^{\|}\left\langle \cos^2 \omega t\right\rangle_t = \tfrac{1}{2}\, p_o u_o^{\|}$$
$$\left\langle I_{a\,i}^{\|\,inst}\left(x=0\right)\right\rangle_t = p_o u_o^{\|}\left\langle \cos\omega t \sin\omega t\right\rangle_t = 0$$

We can define *RMS* pressure and particle velocity amplitudes in terms of their respective ***peak*** amplitudes: $p_o^{rms} \equiv \tfrac{1}{\sqrt{2}}\, p_o$ and $u_o^{\|\,rms} \equiv \tfrac{1}{\sqrt{2}} u_o^{\|}$, then we see that the ***RMS*** active and reactive components of the 1-D longitudinal sound intensity at the listener's position $x = 0$ associated with a 1-D monochromatic plane traveling wave propagating e.g. in the +*x*-direction in "free" air are respectively equal to active and reactive components of the ***time-averaged*** 1-D longitudinal sound intensity at that point, i.e.:

$$I_{a\,r}^{\|\,rms}\left(x=0\right) = \left\langle I_{a\,r}^{\|}\left(x=0\right)\right\rangle_t = \tfrac{1}{2}\, p_o u_o^{\|} = p_o^{rms} u_o^{\|\,rms}$$
$$I_{a\,i}^{\|\,rms}\left(x=0\right) = \left\langle I_{a\,i}^{\|}\left(x=0\right)\right\rangle_t = 0$$

The reader can also easily verify for this example that the ***time-averaged*** active and reactive components of the 1-D longitudinal sound intensity associated with a 1-D monochromatic traveling plane wave propagating e.g. in the +*x*-direction are also correctly given using the complex time-average definition:

$$\left\langle \tilde{I}_a^{\|}\left(x\right)\right\rangle_t \equiv \tfrac{1}{2}\,\tilde{p}\left(x,t\right)\vec{\tilde{u}}^*\left(x,t\right) = \tfrac{1}{2}\, p_o \cancel{e^{i(\omega t - kx)}} u_o^{\|} \cancel{e^{-i(\omega t - kx)}} = p_o u_o^{\|} = \left\langle \tilde{I}_{a\,r}^{\|}\left(x\right)\right\rangle_t + i\left\langle \tilde{I}_{a\,i}^{\|}\left(x\right)\right\rangle_t = p_o u_o^{\|} + 0i$$

Here in this problem, note that $\left\langle \tilde{I}_a^{\|}\left(x\right)\right\rangle_t = \left\langle \tilde{I}_{a\,r}^{\|}\left(x\right)\right\rangle_t + i\left\langle \tilde{I}_{a\,i}^{\|}\left(x\right)\right\rangle_t = p_o u_o^{\|} + 0i = p_o u_o^{\|}$ also has no position (i.e. *x*-) dependence!

The ***instantaneous*** potential, kinetic and total energy densities associated with a 1-D monochromatic traveling plane wave propagating e.g. in the +*x*-direction at $x = 0$ are:

$$w_{potl}^{inst}\left(x=0,t\right) \equiv \frac{1}{2}\frac{1}{\rho_o c^2}\, p_r^2\left(x=0,t\right) = \frac{1}{2}\frac{1}{\rho_o c^2}\, p_o^2 \cos^2 \omega t$$
$$w_{kin}^{inst}\left(x=0,t\right) \equiv \frac{1}{2}\rho_o \vec{u}_r\left(x=0,t\right)\bullet\vec{u}_r\left(x=0,t\right) = \frac{1}{2}\rho_o u_o^{\|2} \cos^2 \omega t$$
$$w_{tot}^{inst}\left(x=0,t\right) \equiv w_{potl}^{inst}\left(x=0,t\right) + w_{kin}^{inst}\left(x=0,t\right) = \frac{1}{2}\frac{1}{\rho_o c^2}\, p_o^2 \cos^2 \omega t + \frac{1}{2}\rho_o u_o^{\|2}\cos^2 \omega t$$

For this situation with a 1-D monochromatic traveling plane wave, we obtained the relation

$$z_a^{\|}\left(x\right) = \frac{p\left(x,t\right)}{u^{\|}\left(x,t\right)} = \frac{p_o}{u_o^{\|}} = \rho_o c \equiv z_o \quad \left(\Omega_a\right)$$

Thus we see again here that: $p_o = \rho_o c u_o^{\|} = z_o u_o^{\|}$. Using the <u>square</u> of this relation in the above expression, we also see that:

$$w_{tot}^{inst}\left(x=0,t\right) \equiv w_{potl}^{inst}\left(x=0,t\right) + w_{kin}^{inst}\left(x=0,t\right) = \frac{1}{\rho_o c^2}\, p_o^2 \cos^2 \omega t = \rho_o u_o^{\|2} \cos^2 \omega t$$

The ***time-average*** of the potential, kinetic and total energy densities associated with a 1-D monochromatic traveling plane wave propagating e.g. in the +*x*-direction are:

$$\left\langle w_{potl}\left(x\right)\right\rangle_t \equiv \frac{1}{4}\frac{\left|\tilde{p}\left(x,t\right)\right|^2}{\rho_o c^2} = \frac{1}{4}\frac{1}{\rho_o c^2}\, p_o^2 \quad \left(Joules/m^3\right)$$

$$\left\langle w_{kin}\left(x\right)\right\rangle_t \equiv \frac{1}{4}\rho_o\left(\vec{\tilde{u}}\left(x,t\right)\bullet\vec{\tilde{u}}^*\left(x,t\right)\right) = \frac{1}{4}\rho_o\left|\vec{\tilde{u}}\left(x,t\right)\right|^2 = \frac{1}{4}\rho_o u_o^{\parallel 2} \qquad \left(Joules/m^3\right)$$

$$\left\langle w_{tot}\left(x\right)\right\rangle_t \equiv \left\langle w_{potl}\left(x\right)\right\rangle_t + \left\langle w_{kin}\left(x\right)\right\rangle_t = \frac{1}{4}\frac{1}{\rho_o c^2}\, p_o^2 + \frac{1}{4}\rho_o u_o^{\parallel 2} \quad \left(Joules/m^3\right)$$

Again, using the square of the relation $p_o = \rho_o c u_o^{\parallel} = z_o u_o^{\parallel}$ in the above expression, we see that:

$$\left\langle w_{tot}\left(x\right)\right\rangle_t \equiv \left\langle w_{potl}\left(x\right)\right\rangle_t + \left\langle w_{kin}\left(x\right)\right\rangle_t = \frac{1}{2}\frac{1}{\rho_o c^2}\, p_o^2 = \frac{1}{2}\rho_o u_o^{\parallel 2} = \frac{1}{2c}\, p_o u_o^{\parallel} \quad \left(Joules/m^3\right)$$

Note that the ratio of the ***time-averaged*** potential energy density to the ***time-averaged*** kinetic energy density e.g. at *x* = 0 is equal to <u>unity</u> for a 1-D monochromatic traveling wave:

$$\frac{\left\langle w_{potl}\left(x\right)\right\rangle_t}{\left\langle w_{kin}\left(x\right)\right\rangle_t} = \frac{\frac{1}{4}\frac{\left|\tilde{p}\left(x,t\right)\right|^2}{\rho_o c^2}}{\frac{1}{4}\rho_o\left|\vec{\tilde{u}}\left(x,t\right)\right|^2} = \frac{\frac{1}{4}\frac{1}{\rho_o c^2}\, p_o^2}{\frac{1}{4}\rho_o u_o^{\parallel 2}} = \frac{p_o^2}{\rho_o^2 c^2 u_o^{\parallel 2}} = \frac{p_o^2}{z_o^2 u_o^{\parallel 2}} = \frac{z_o^2}{z_o^2} = 1$$

Note further that:

$$\left\langle \tilde{I}_a^{\parallel}\left(x\right)\right\rangle_t = \frac{1}{2}\, p_o u_o^{\parallel} = \frac{1}{2}\frac{1}{\rho_o c}\, p_o^2 = \frac{1}{2}\rho_o c u_o^{\parallel 2} \quad \left(Watts/m^2\right)$$

Again using the relation $p_o = \rho_o c u_o^{\parallel} = z_o u_o^{\parallel}$, we also see that for a 1-D monochromatic traveling plane wave that:

$$\left\langle \tilde{I}_a^{\parallel}\left(x\right)\right\rangle_t = c\left\langle w_{tot}\left(x\right)\right\rangle_t = \frac{1}{2}\, p_o u_o^{\parallel} = \frac{1}{2}\frac{1}{\rho_o c}\, p_o^2 = \frac{1}{2}\rho_o c u_o^{\parallel 2} \quad \left(Watts/m^2\right)$$

Example # 2: Two Counter-Propagating 1-D Plane Monochromatic Traveling Waves in "Free Air":

In this example, we imagine two un-equal strength harmonic (i.e. single-frequency) sound sources located at $x = \pm\infty$, with an observer/listener located near/at the origin $x = 0$. At the observer's location there will therefore be two 1-D monochromatic plane traveling waves propagating in opposite directions in "free" air (i.e. the Great Wide-Open).

The physical over-pressure amplitudes associated with the right- and left-going monochromatic plane waves are purely real quantities:

$$p_A(x,t) = A\cos(\omega t - kx + \varphi_A) \text{ and } p_B(x,t) = B\cos(\omega t + kx + \varphi_B) \text{ with } A \neq B \text{ \{necessarily\}}$$

Using complex notation, the individual over-pressure amplitudes are:

$$\tilde{p}_A(x,t) = \tilde{A}e^{i(\omega t - kx)} \text{ and } \tilde{p}_B(x,t) = \tilde{B}e^{i(\omega t + kx)} \text{ with } \tilde{A} \neq \tilde{B} \text{ \{necessarily\}}$$

where $\tilde{A} = |\tilde{A}|e^{i\varphi_A} = Ae^{i\varphi_A}$ and $\tilde{B} = |\tilde{B}|e^{i\varphi_B} = Be^{i\varphi_A}$.

Since each individual complex over-pressure amplitude satisfies its own Euler's equation:

$$\frac{\partial u^{\parallel}_{A,B}(x,t)}{\partial t} = -\frac{1}{\rho_o}\frac{\partial p_{A,B}(x,t)}{\partial x}$$

the corresponding right- and left-going 1-D longitudinal complex particle velocities are:

$$\tilde{u}^{\parallel}_A(x,t) = \frac{\tilde{A}}{\rho_o c}e^{i(\omega t - kx)} \equiv \tilde{u}^{\parallel}_{A_o}e^{i(\omega t - kx)} \text{ and } \tilde{u}^{\parallel}_B(x,t) = -\frac{\tilde{B}}{\rho_o c}e^{i(\omega t + kx)} \equiv -\tilde{u}^{\parallel}_{B_o}e^{i(\omega t + kx)} \text{ (using } c = \omega/k\text{)}$$

Note the –*ve* sign in the left-going 1-D longitudinal complex particle velocity amplitude, which simple reflects the fact that it is propagating in the –*ve* *x*-direction.

For sound pressure levels $SPL = L_p = 20\log_{10}(p_{atm}/p_o) \ll 134\, dB$, corresponding to sound over-pressure amplitudes in "free" air at NTP of $|\tilde{p}(\vec{r},t)| \ll 100\ RMS\ Pascals$, the principle of linear superposition holds, such that the total/resultant complex over-pressure and longitudinal particle velocity amplitudes respectively are:

$$\tilde{p}_{tot}(x,t) = \tilde{p}_A(x,t) + \tilde{p}_B(x,t) = \tilde{A}e^{i(\omega t - kx)} + \tilde{B}e^{i(\omega t + kx)} \quad (Pascals)$$

and:

$$\tilde{u}^{\parallel}_{tot}(x,t) = \tilde{u}^{\parallel}_A(x,t) + \tilde{u}^{\parallel}_B(x,t) = \tilde{u}^{\parallel}_{A_o}e^{i(\omega t - kx)} + \tilde{u}^{\parallel}_{B_o}e^{i(\omega t + kx)} = \frac{\tilde{A}}{\rho_o c}e^{i(\omega t - kx)} - \frac{\tilde{B}}{\rho_o c}e^{i(\omega t + kx)} \quad (m/s)$$

We can recast the above equations in terms of the dimensionless complex variable:

$$\tilde{R} \equiv \frac{\tilde{B}}{\tilde{A}} = \frac{|\tilde{B}|e^{i\varphi_B}}{|\tilde{A}|e^{i\varphi_A}} = \frac{|\tilde{B}|}{|\tilde{A}|}e^{i(\varphi_B-\varphi_A)} = |\tilde{R}|e^{i(\varphi_B-\varphi_A)} = |\tilde{R}|e^{i\Delta\varphi_{BA}}$$

Thus:

$$\tilde{p}_{tot}(x,t) = \tilde{A}\left[e^{i(\omega t-kx)} + |\tilde{R}|e^{i(\omega t+kx)}\cdot e^{i\Delta\varphi_{BA}}\right] = \tilde{A}e^{i(\omega t-kx)}\left[1 + |\tilde{R}|e^{i(2kx+\Delta\varphi_{BA})}\right]$$

and:

$$\tilde{u}_{tot}^{\parallel}(x,t) = \frac{\tilde{A}}{\rho_o c}\left[e^{i(\omega t-kx)} - |\tilde{R}|e^{i(\omega t+kx)}\cdot e^{i\Delta\varphi_{BA}}\right] = \frac{\tilde{A}}{\rho_o c}e^{i(\omega t-kx)}\left[1 - |\tilde{R}|e^{i(2kx+\Delta\varphi_{BA})}\right]$$

We first calculate the magnitudes of the complex total/resultant over-pressure $|\tilde{p}_{tot}(x,t)|$ and 1-D longitudinal particle velocity $|\tilde{u}_{tot}^{\parallel}(x,t)|$:

$$\begin{aligned}
|\tilde{p}_{tot}(x,t)| &\equiv \sqrt{\tilde{p}_{tot}(x,t)\cdot\tilde{p}_{tot}^{*}(x,t)} \\
&= |\tilde{A}|\sqrt{\left(1 + |\tilde{R}|e^{i(2kx+\Delta\varphi_{BA})}\right)\cdot\left(1 + |\tilde{R}|e^{i(2kx+\Delta\varphi_{BA})}\right)^{*}} \\
&= |\tilde{A}|\sqrt{\left(1 + |\tilde{R}|e^{i(2kx+\Delta\varphi_{BA})}\right)\cdot\left(1 + |\tilde{R}|e^{-i(2kx+\Delta\varphi_{BA})}\right)} \\
&= |\tilde{A}|\sqrt{1 + |\tilde{R}|e^{i(2kx+\Delta\varphi_{BA})} + |\tilde{R}|e^{-i(2kx+\Delta\varphi_{BA})} + |\tilde{R}|^2} \\
&= |\tilde{A}|\sqrt{1 + |\tilde{R}|\left\{e^{i(2kx+\Delta\varphi_{BA})} + e^{-i(2kx+\Delta\varphi_{BA})}\right\} + |\tilde{R}|^2} \\
&= |\tilde{A}|\sqrt{1 + 2|\tilde{R}|\cos(2kx+\Delta\varphi_{BA}) + |\tilde{R}|^2}
\end{aligned}$$

and:

$$\begin{aligned}
|\tilde{u}_{tot}^{\parallel}(x,t)| &\equiv \sqrt{\tilde{u}_{tot}^{\parallel}(x,t)\cdot\tilde{u}_{tot}^{\parallel *}(x,t)} \\
&= \frac{|\tilde{A}|}{\rho_o c}\sqrt{\left(1 - |\tilde{R}|e^{i(2kx+\Delta\varphi_{BA})}\right)\cdot\left(1 - |\tilde{R}|e^{i(2kx+\Delta\varphi_{BA})}\right)^{*}} \\
&= \frac{|\tilde{A}|}{\rho_o c}\sqrt{\left(1 - |\tilde{R}|e^{i(2kx+\Delta\varphi_{BA})}\right)\cdot\left(1 - |\tilde{R}|e^{-i(2kx+\Delta\varphi_{BA})}\right)} \\
&= \frac{|\tilde{A}|}{\rho_o c}\sqrt{1 - |\tilde{R}|e^{i(2kx+\Delta\varphi_{BA})} - |\tilde{R}|e^{-i(2kx+\Delta\varphi_{BA})} + |\tilde{R}|^2} \\
&= \frac{|\tilde{A}|}{\rho_o c}\sqrt{1 - |\tilde{R}|\left\{e^{i(2kx+\Delta\varphi_{BA})} + e^{-i(2kx+\Delta\varphi_{BA})}\right\} + |\tilde{R}|^2} \\
&= \frac{|\tilde{A}|}{\rho_o c}\sqrt{1 - 2|\tilde{R}|\cos(2kx+\Delta\varphi_{BA}) + |\tilde{R}|^2}
\end{aligned}$$

Thus, e.g. for an observer/listener's position $x = 0$, and for ***equal-strength*** pressure amplitudes $|\tilde{A}| = |\tilde{B}| \Rightarrow |\tilde{R}| \equiv |\tilde{B}|/|\tilde{A}| = 1$ (i.e. a pure standing wave!) these formulae simplify to:

$$|\tilde{p}_{tot}(x=0,t)| = \sqrt{2}|\tilde{A}|\sqrt{1+\cos\Delta\varphi_{BA}} \text{ and: } |\tilde{u}^{\parallel}_{tot}(x=0,t)| = \frac{\sqrt{2}|\tilde{A}|}{\rho_o c}\sqrt{1-\cos\Delta\varphi_{BA}}$$

Thus, we see that when: $\Delta\varphi_{BA} = 0, \pm 2\pi, \pm 4\pi, ... = \pm n_{even}\pi$ that: $\cos\Delta\varphi_{BA} = +1$ and thus:

$$|\tilde{p}_{tot}(x=0,t)| = 2|\tilde{A}| \text{ and: } |\tilde{u}^{\parallel}_{tot}(x=0,t)| = 0$$

i.e. we have complete constructive (destructive) interference associated with the two individual complex over-pressure (1-D longitudinal particle velocity) amplitudes, respectively.

We also see that when: $\Delta\varphi_{BA} = \pm 1\pi, \pm 3\pi, \pm 5\pi, ... = \pm n_{odd}\pi$ that: $\cos\Delta\varphi_{BA} = -1$ and thus:

$$|\tilde{p}_{tot}(x=0,t)| = 0 \text{ and: } |\tilde{u}^{\parallel}_{tot}(x=0,t)| = \frac{2|\tilde{A}|}{\rho_o c}$$

i.e. we have complete destructive (constructive) interference associated with the two individual complex over-pressure (1-D longitudinal particle velocity) amplitudes, respectively.

Hence, we can also now see that when $|\tilde{R}| \equiv |\tilde{B}|/|\tilde{A}| \neq 1$, it is *not* possible to ever achieve *complete* constructive/destructive interference effects between the two individual right- and left-moving complex over-pressure and/or 1-D longitudinal particle velocity amplitudes.

Since $\tilde{p}_{tot}(x,t) = |\tilde{p}_{tot}(x,t)|e^{i\varphi_p(x,t)}$ and $\tilde{u}^{\parallel}_{tot}(x,t) = |\tilde{u}^{\parallel}_{tot}(x,t)|e^{i\varphi_u(x,t)}$, the phases of the complex total/resultant pressure and 1-D longitudinal particle velocity associated with the two counter-propagating 1-D monochromatic plane waves are given by:

$$\varphi_{p_{tot}}(x,t) \equiv \tan^{-1}\left(\frac{p_{tot\,i}(x,t)}{p_{tot\,r}(x,t)}\right) = \textit{big mess}!$$

and:

$$\varphi_{u_{tot}}(x,t) \equiv \tan^{-1}\left(\frac{u^{\parallel}_{tot\,i}(x,t)}{u^{\parallel}_{tot\,r}(x,t)}\right) = \textit{big mess}!$$

We will work these out later… Below, we will see that the expression for the phase difference $\Delta\varphi_{p_{tot}-u_{tot}}(x) \equiv \varphi_{p_{tot}}(x) - \varphi_{u_{tot}}(x) = \varphi_z(x) = \varphi_I(x)$ will be much simpler.

The complex 1-D longitudinal *specific* acoustic impedance associated with the two counter-propagating 1-D monochromatic plane waves is:

$$\tilde{z}^{\parallel}_{a\,tot}(x) \equiv \frac{\tilde{p}_{tot}(x,t)}{\tilde{u}^{\parallel}_{tot}(x,t)} = \frac{\cancel{\tilde{A}}\,\cancel{e^{i(\omega t-kx)}}\left[1+|\tilde{R}|e^{i(2kx+\Delta\varphi_{BA})}\right]}{\frac{\cancel{\tilde{A}}}{\rho_o c}\cancel{e^{i(\omega t-kx)}}\left[1-|\tilde{R}|e^{i(2kx+\Delta\varphi_{BA})}\right]} = \rho_o c\frac{\left[1+|\tilde{R}|e^{i(2kx+\Delta\varphi_{BA})}\right]}{\left[1-|\tilde{R}|e^{i(2kx+\Delta\varphi_{BA})}\right]}$$

Since the ***characteristic*** *specific* acoustic impedance of "free" air is $z_o \equiv \rho_o c$, then:

$$\tilde{z}_{a\,tot}^{\parallel}(x) = z_o \frac{\left[1 + |\tilde{R}| e^{i(2kx+\Delta\varphi_{BA})}\right]}{\left[1 - |\tilde{R}| e^{i(2kx+\Delta\varphi_{BA})}\right]} = z_o \frac{\left[1 + |\tilde{R}| e^{i(2kx+\Delta\varphi_{BA})}\right]}{\left[1 - |\tilde{R}| e^{i(2kx+\Delta\varphi_{BA})}\right]} \cdot \frac{\left[1 - |\tilde{R}| e^{i(2kx+\Delta\varphi_{BA})}\right]^*}{\left[1 - |\tilde{R}| e^{i(2kx+\Delta\varphi_{BA})}\right]^*}$$

$$= z_o \frac{\left[1 + |\tilde{R}| e^{i(2kx+\Delta\varphi_{BA})}\right]}{\left[1 - |\tilde{R}| e^{i(2kx+\Delta\varphi_{BA})}\right]} \cdot \frac{\left[1 - |\tilde{R}| e^{-i(2kx+\Delta\varphi_{BA})}\right]}{\left[1 - |\tilde{R}| e^{-i(2kx+\Delta\varphi_{BA})}\right]} = z_o \frac{\left[1 + |\tilde{R}| e^{i(2kx+\Delta\varphi_{BA})} - |\tilde{R}| e^{-i(2kx+\Delta\varphi_{BA})} - |\tilde{R}|^2\right]}{\left[1 - |\tilde{R}| e^{i(2kx+\Delta\varphi_{BA})} - |\tilde{R}| e^{-i(2kx+\Delta\varphi_{BA})} + |\tilde{R}|^2\right]}$$

$$= z_o \frac{\left[1 + |\tilde{R}| \left\{e^{i(2kx+\Delta\varphi_{BA})} - e^{-i(2kx+\Delta\varphi_{BA})}\right\} - |\tilde{R}|^2\right]}{\left[1 - |\tilde{R}| \left\{e^{i(2kx+\Delta\varphi_{BA})} + e^{-i(2kx+\Delta\varphi_{BA})}\right\} + |\tilde{R}|^2\right]} = z_o \frac{\left[\left\{1 - |\tilde{R}|^2\right\} + |\tilde{R}| \left\{e^{i(2kx+\Delta\varphi_{BA})} - e^{-i(2kx+\Delta\varphi_{BA})}\right\}\right]}{\left[\left\{1 + |\tilde{R}|^2\right\} - |\tilde{R}| \left\{e^{i(2kx+\Delta\varphi_{BA})} + e^{-i(2kx+\Delta\varphi_{BA})}\right\}\right]}$$

Using the relations $\cos\theta \equiv \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$ and $\sin\theta \equiv \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$:

$$\tilde{z}_{a\,tot}^{\parallel}(x) = z_o \frac{\left[\left\{1 - |\tilde{R}|^2\right\} + 2i|\tilde{R}|\sin(2kx + \Delta\varphi_{BA})\right]}{\left[\left\{1 + |\tilde{R}|^2\right\} - 2|\tilde{R}|\cos(2kx + \Delta\varphi_{BA})\right]}$$

Note that $\tilde{z}_{tot}^{\parallel}(x)$ has *no* explicit time dependence, but certainly does have spatial/position (*x*-) and frequency (*f*-) dependence (via the wavenumber $k = 2\pi/\lambda = 2\pi f/c$)!

The magnitude of the complex 1-D longitudinal specific acoustic impedance associated with the two counter-propagating 1-D monochromatic plane waves is:

$$\left|\tilde{z}_{a\,tot}^{\parallel}(x)\right| \equiv \sqrt{\tilde{z}_{tot}^{\parallel}(x) \cdot \tilde{z}_{tot}^{\parallel *}(x)}$$

$$= z_o \frac{\sqrt{\left[\left\{1 - |\tilde{R}|^2\right\} + 2i|\tilde{R}|\sin(2kx + \Delta\varphi_{BA})\right] \cdot \left[\left\{1 - |\tilde{R}|^2\right\} + 2i|\tilde{R}|\sin(2kx + \Delta\varphi_{BA})\right]^*}}{\left[\left\{1 + |\tilde{R}|^2\right\} - 2|\tilde{R}|\cos(2kx + \Delta\varphi_{BA})\right]}$$

$$= z_o \frac{\sqrt{\left[\left\{1 - |\tilde{R}|^2\right\} + 2i|\tilde{R}|\sin(2kx + \Delta\varphi_{BA})\right] \cdot \left[\left\{1 - |\tilde{R}|^2\right\} - 2i|\tilde{R}|\sin(2kx + \Delta\varphi_{BA})\right]}}{\left[\left\{1 + |\tilde{R}|^2\right\} - 2|\tilde{R}|\cos(2kx + \Delta\varphi_{BA})\right]}$$

$$= z_o \frac{\sqrt{\left\{1 - |\tilde{R}|^2\right\}^2 + 4|\tilde{R}|^2\sin^2(2kx + \Delta\varphi_{BA})}}{\left[\left\{1 + |\tilde{R}|^2\right\} - 2|\tilde{R}|\cos(2kx + \Delta\varphi_{BA})\right]}$$

The phase of the complex 1-D longitudinal *specific* acoustic impedance associated with the two counter-propagating 1-D monochromatic plane waves is:

$$\varphi_z(x) \equiv \tan^{-1}\left(\frac{z^{\|}_{a\,tot\,i}(x)}{z^{\|}_{a\,tot\,r}(x)}\right) = \tan^{-1}\left(\frac{\cancel{z_o}\left[2|\tilde{R}|\sin(2kx+\Delta\varphi_{BA})\Big/\cancel{\left[\left\{1+|\tilde{R}|^2\right\}-2|\tilde{R}|\cos(2kx+\Delta\varphi_{BA})\right]}\right]}{\cancel{z_o}\left[\left\{1-|\tilde{R}|^2\right\}\Big/\cancel{\left[\left\{1+|\tilde{R}|^2\right\}-2|\tilde{R}|\cos(2kx+\Delta\varphi_{BA})\right]}\right]}\right)$$

$$= \tan^{-1}\left(\frac{2|\tilde{R}|\sin(2kx+\Delta\varphi_{BA})}{\left\{1-|\tilde{R}|^2\right\}}\right) = \Delta\varphi_{p_{tot}-u_{tot}}(x) = \varphi_{p_{tot}}(x) - \varphi_{u_{tot}}(x)$$

Thus, e.g. for an observer/listener's position $x = 0$, and for *equal-strength* pressure amplitudes $|\tilde{A}| = |\tilde{B}| \;\Rightarrow\; |\tilde{R}| \equiv |\tilde{B}|/|\tilde{A}| = 1$ (i.e. a "pure" standing wave) these two formulae simplify to:

$$\tilde{z}^{\|}_{a\,tot}(x=0) = z_o \frac{i\sin\Delta\varphi_{BA}}{\left[1-\cos\Delta\varphi_{BA}\right]}$$

and:

$$\varphi_z(0) = \tan^{-1}\left(\frac{2\sin\Delta\varphi_{BA}}{0}\right) = \tan^{-1}(\pm\infty) = \Delta\varphi_{p_{tot}-u_{tot}}(0) = \varphi_{p_{tot}}(0) - \varphi_{u_{tot}}(0)$$
$$= \pm\pi/2, \pm 3\pi/2, \pm 5\pi/2, ... \;\; = \pm n_{odd}\,\pi/2$$

i.e. for an observer/listener's position $x = 0$, and for *equal-strength* pressure amplitudes $|\tilde{A}| = |\tilde{B}|$ $\Rightarrow |\tilde{R}| \equiv |\tilde{B}|/|\tilde{A}| = 1$ the complex 1-D longitudinal *specific* acoustic impedance $\tilde{z}^{\|}_{tot}(x=0)$ is purely *imaginary*; its phase $\varphi_z(x=0)$ is an odd integer multiple of $\pm\pi/2 = \pm 90^{\text{o}}$ – which in turn also tells us that in this situation, the complex pressure $\tilde{p}_{tot}(x=0,t)$ and 1-D longitudinal particle velocity $\tilde{u}^{\|}_{tot}(x=0,t)$ differ in phase by an *odd* integer multiple of $\pm\pi/2 = \pm 90^{\text{o}}$.

Note that in general, *maxima* associated with the complex 1-D longitudinal specific acoustic impedance $\tilde{z}^{\|}_{tot}(x)$ for this situation occur whenever $2kx + \Delta\varphi_{BA} = 0, \pm 2\pi, \pm 4\pi, \pm 6\pi ... = \pm n_{even}\pi$, i.e. whenever $\cos(2kx+\Delta\varphi_{BA}) = +1$, and $\sin(2kx+\Delta\varphi_{BA}) = 0$, then:

$$\left|\tilde{z}^{\|}_{a\,tot}(x)\right|_{\text{maxima}} = z_o \frac{\sqrt{\left\{1-|\tilde{R}|^2\right\}^2 + 4|\tilde{R}|^2\sin^2(2kx+\Delta\varphi_{BA})}}{\left[\left\{1+|\tilde{R}|^2\right\}-2|\tilde{R}|\cos(2kx+\Delta\varphi_{BA})\right]} = z_o \frac{\sqrt{\left\{1-|\tilde{R}|^2\right\}^2}}{\left[\left\{1+|\tilde{R}|^2\right\}-2|\tilde{R}|\right]}$$

$$= z_o\left(\frac{1-|\tilde{R}|^2}{1-2|\tilde{R}|+|\tilde{R}|^2}\right) = z_o\left(\frac{1-|\tilde{R}|^2}{\left(1-|\tilde{R}|\right)^2}\right) = z_o \frac{\cancel{\left(1-|\tilde{R}|\right)}\cdot\left(1+|\tilde{R}|\right)}{\cancel{\left(1-|\tilde{R}|\right)}\cdot\left(1-|\tilde{R}|\right)} = z_o\frac{\left(1+|\tilde{R}|\right)}{\left(1-|\tilde{R}|\right)}$$

The phase associated with the complex 1-D longitudinal *specific* acoustic impedance *maxima* is:

$$\varphi_z(x)\Big|_{\text{maxima}} = \tan^{-1}\left(\frac{2\left|\tilde{R}\right|\sin\left(2kx+\Delta\varphi_{BA}\right)}{\left\{1-\left|\tilde{R}\right|^2\right\}}\right) = \tan^{-1}(0) = 0$$

$$= \Delta\varphi_{p_{tot}-u_{tot}}(x)\Big|_{\text{maxima}} = \left(\varphi_{p_{tot}}(x) - \varphi_{u_{tot}}(x)\right)_{\text{maxima}}$$

Thus, for 1-D longitudinal *specific* acoustic impedance *maxima* associated with this situation, we see that the total/resultant complex pressure $\tilde{p}_{tot}(x,t)$ and 1-D longitudinal particle velocity $\tilde{u}^{\parallel}_{tot}(x,t)$ are precisely *in-phase* with each other, or at least by ± *even* integer multiples of pi.

Since $\left|\tilde{z}^{\parallel}_{tot}(x)\right|_{\text{maxima}} = \left|\tilde{p}_{tot}(x,t)\right| / \left|\tilde{u}^{\parallel}_{tot}(x,t)\right|_{\text{maxima}}$ this also tells us that whenever $\left(2kx+\Delta\varphi_{BA}\right) = 0, \pm 2\pi, \pm 4\pi, \pm 6\pi ... = \pm n_{even}\pi$, the magnitude of the total/resultant complex pressure $\left|\tilde{p}_{tot}(x,t)\right|$ will also be a *maxima*, whereas the magnitude of the total/resultant 1-D complex longitudinal particle velocity $\left|\tilde{u}^{\parallel}_{tot}(x,t)\right|$ will simultaneously be a *minima*:

$$\tilde{p}_{tot}(x,t) = \tilde{A}e^{i(\omega t - kx)}\left[1 + \left|\tilde{R}\right|e^{i(2kx+\Delta\varphi_{BA})}\right] = \tilde{A}e^{i(\omega t - kx)}\left[1 + \left|\tilde{R}\right|e^{\pm i n_{even}\pi}\right]$$

$$= \tilde{A}e^{i(\omega t - kx)}\left[1 + \left|\tilde{R}\right|\left\{\cos\left(\pm n_{even}\pi\right) + \cancel{i\sin\left(\pm n_{even}\pi\right)}\right\}\right] = \tilde{A}e^{i(\omega t - kx)}\left[1 + \left|\tilde{R}\right|\right]$$

and:

$$\tilde{u}^{\parallel}_{tot}(x,t) = \frac{\tilde{A}}{\rho_o c}e^{i(\omega t - kx)}\left[1 - \left|\tilde{R}\right|e^{i(2kx+\Delta\varphi_{BA})}\right] = \frac{\tilde{A}}{\rho_o c}e^{i(\omega t - kx)}\left[1 - \left|\tilde{R}\right|e^{\pm i n_{even}\pi}\right]$$

$$= \frac{\tilde{A}}{\rho_o c}e^{i(\omega t - kx)}\left[1 - \left|\tilde{R}\right|\left\{\cos\left(\pm n_{even}\pi\right) + \cancel{i\sin\left(\pm n_{even}\pi\right)}\right\}\right] = \frac{\tilde{A}}{\rho_o c}e^{i(\omega t - kx)}\left[1 - \left|\tilde{R}\right|\right]$$

$$\Rightarrow \quad \left|\tilde{p}_{tot}(x,t)\right| \equiv \sqrt{\tilde{p}_{tot}(x,t)\cdot\tilde{p}^{*}_{tot}(x,t)} = \left|\tilde{A}\right|\left[1 + \left|\tilde{R}\right|\right] \qquad \text{for } \left|\tilde{R}\right| = 1: \left|\tilde{p}_{tot}(x,t)\right| = 2\left|\tilde{A}\right|$$

$$\Rightarrow \quad \left|\tilde{u}^{\parallel}_{tot}(x,t)\right| \equiv \sqrt{\tilde{u}^{\parallel}_{tot}(x,t)\cdot\tilde{u}^{\parallel *}_{tot}(x,t)} = \frac{\left|\tilde{A}\right|}{\rho_o c}\left[1 - \left|\tilde{R}\right|\right] \qquad \text{for } \left|\tilde{R}\right| = 1: \left|\tilde{u}^{\parallel}_{tot}(x,t)\right| = 0$$

"Pure" standing wave!!!

In general, *minima* associated with the complex 1-D longitudinal specific acoustic impedance $\tilde{z}^{\parallel}_{tot}(x)$ for this situation will occur whenever $\left(2kx+\Delta\varphi_{BA}\right) = \pm 1\pi, \pm 3\pi, \pm 5\pi ... = \pm n_{odd}\pi$, i.e. whenever $\cos\left(2kx+\Delta\varphi_{BA}\right) = -1$, and $\sin\left(2kx+\Delta\varphi_{BA}\right) = 0$, then:

$$\left|\tilde{z}^{\parallel}_{a\,tot}(x)\right|_{\text{minima}} = z_o \frac{\sqrt{\left\{1-\left|\tilde{R}\right|^2\right\}^2 + 4\left|\tilde{R}\right|^2 \sin^2\left(2kx+\Delta\varphi_{BA}\right)}}{\left[\left\{1+\left|\tilde{R}\right|^2\right\} - 2\left|\tilde{R}\right|\cos\left(2kx+\Delta\varphi_{BA}\right)\right]} = z_o \frac{\sqrt{\left\{1-\left|\tilde{R}\right|^2\right\}^2}}{\left[\left\{1+\left|\tilde{R}\right|^2\right\} + 2\left|\tilde{R}\right|\right]}$$

$$= z_o \left(\frac{1-\left|\tilde{R}\right|^2}{1+2\left|\tilde{R}\right|+\left|\tilde{R}\right|^2}\right) = z_o \left(\frac{1-\left|\tilde{R}\right|^2}{\left(1+\left|\tilde{R}\right|\right)^2}\right) = z_o \frac{\left(1-\left|\tilde{R}\right|\right)\cdot\cancel{\left(1+\left|\tilde{R}\right|\right)}}{\left(1+\left|\tilde{R}\right|\right)\cdot\cancel{\left(1+\left|\tilde{R}\right|\right)}} = z_o \frac{\left(1-\left|\tilde{R}\right|\right)}{\left(1+\left|\tilde{R}\right|\right)}$$

The phase associated with the complex 1-D longitudinal *specific* acoustic impedance *minima* is:

$$\varphi_z(x)\Big|_{\text{minima}} = \tan^{-1}\left(\frac{2\left|\tilde{R}\right|\sin\left(2kx+\Delta\varphi_{BA}\right)}{\left\{1-\left|\tilde{R}\right|^2\right\}}\right) = \tan^{-1}(0) = 0$$

$$= \Delta\varphi_{p_{tot}-u_{tot}}(x)\Big|_{\text{minima}} = \left(\varphi_{p_{tot}}(x) - \varphi_{u_{tot}}(x)\right)_{\text{minima}}$$

Thus, for 1-D longitudinal *specific* acoustic impedance *minima* associated with this situation, we see that the total/resultant complex pressure $\tilde{p}_{tot}(x,t)$ and 1-D longitudinal particle velocity $\tilde{u}^{\parallel}_{tot}(x,t)$ are precisely *out-of-phase* with each other, or at least by ± *odd* integer multiples of pi.

Since $\left|\tilde{z}^{\parallel}_{tot}(x)\right|_{\text{minima}} = \left|\tilde{p}_{tot}(x,t)\right| / \left|\tilde{u}^{\parallel}_{tot}(x,t)\right|_{\text{minima}}$ this also tells us that whenever $2kx + \Delta\varphi_{BA} = \pm 1\pi, \pm 3\pi, \pm 5\pi ... = \pm n_{odd}\pi$, the magnitude of the total/resultant complex pressure $\left|\tilde{p}_{tot}(x,t)\right|$ will also be a *minima*, whereas the magnitude of the total/resultant 1-D complex longitudinal particle velocity $\left|\tilde{u}^{\parallel}_{tot}(x,t)\right|$ will simultaneously be a *maxima*:

$$\tilde{p}_{tot}(x,t) = \tilde{A}e^{i(\omega t-kx)}\left[1 + \left|\tilde{R}\right|e^{i(2kx+\Delta\varphi_{BA})}\right] = \tilde{A}e^{i(\omega t-kx)}\left[1 + \left|\tilde{R}\right|e^{\pm i n_{odd}\pi}\right]$$

$$= \tilde{A}e^{i(\omega t-kx)}\left[1 + \left|\tilde{R}\right|\left\{\cos\left(\pm n_{odd}\pi\right) + \cancel{i\sin\left(\pm n_{odd}\pi\right)}\right\}\right] = \tilde{A}e^{i(\omega t-kx)}\left[1 - \left|\tilde{R}\right|\right]$$

and:

$$\tilde{u}^{\parallel}_{tot}(x,t) = \frac{\tilde{A}}{\rho_o c}e^{i(\omega t-kx)}\left[1 - \left|\tilde{R}\right|e^{i(2kx+\Delta\varphi_{BA})}\right] = \frac{\tilde{A}}{\rho_o c}e^{i(\omega t-kx)}\left[1 - \left|\tilde{R}\right|e^{\pm i n_{odd}\pi}\right]$$

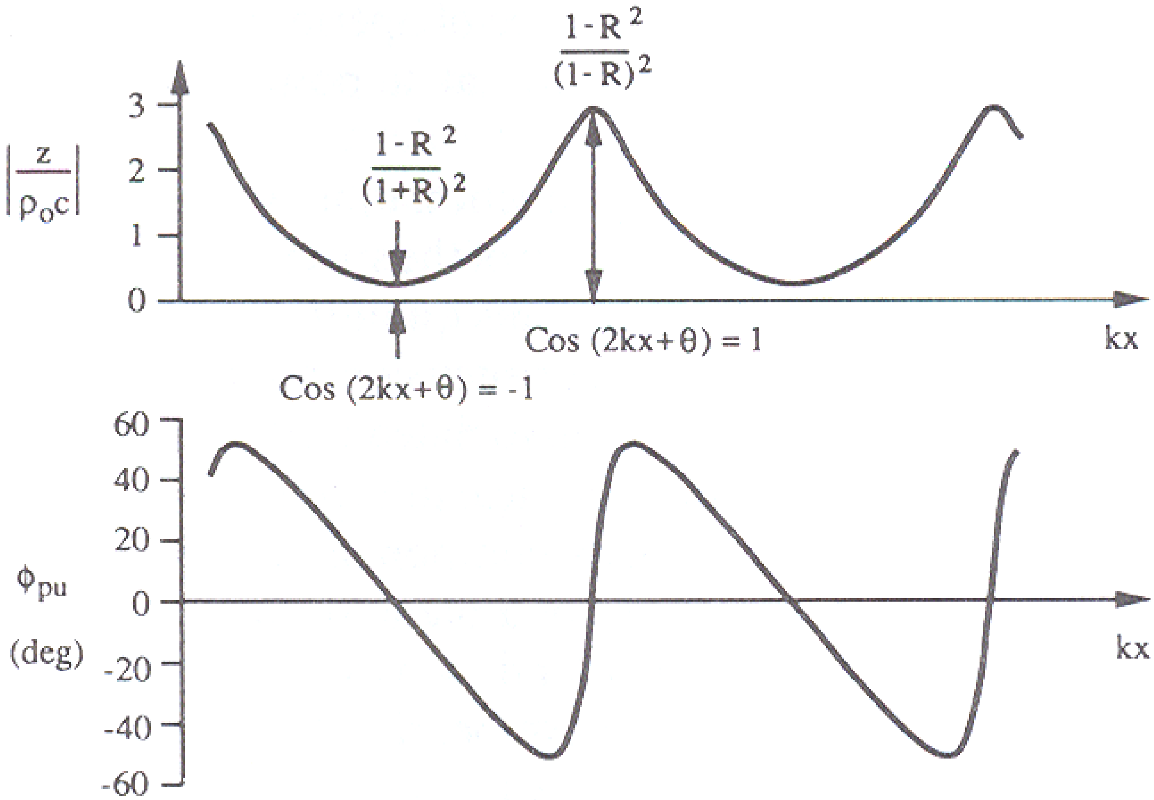
$$= \frac{\tilde{A}}{\rho_o c}e^{i(\omega t-kx)}\left[1 - \left|\tilde{R}\right|\left\{\cos\left(\pm n_{odd}\pi\right) + \cancel{i\sin\left(\pm n_{odd}\pi\right)}\right\}\right] = \frac{\tilde{A}}{\rho_o c}e^{i(\omega t-kx)}\left[1 + \left|\tilde{R}\right|\right]$$

$$\Rightarrow \quad \left|\tilde{p}_{tot}(x,t)\right| \equiv \sqrt{\tilde{p}_{tot}(x,t)\cdot\tilde{p}^{*}_{tot}(x,t)} = \left|\tilde{A}\right|\left[1 - \left|\tilde{R}\right|\right] \qquad \text{for } \left|\tilde{R}\right| = 1: \ \left|\tilde{p}_{tot}(x,t)\right| = 0$$

$$\Rightarrow \quad \left|\tilde{u}^{\parallel}_{tot}(x,t)\right| \equiv \sqrt{\tilde{u}^{\parallel}_{tot}(x,t)\cdot\tilde{u}^{\parallel *}_{tot}(x,t)} = \frac{\left|\tilde{A}\right|}{\rho_o c}\left[1 + \left|\tilde{R}\right|\right] \qquad \text{for } \left|\tilde{R}\right| = 1: \ \left|\tilde{u}^{\parallel}_{tot}(x,t)\right| = \frac{2\left|\tilde{A}\right|}{\rho_o c}$$

"Pure" standing wave!!!

A perhaps somewhat more general situation associated with two counter-propagating monochromatic plane waves in “free air”, is e.g. the case when $|\tilde{R}| \equiv |\tilde{B}|/|\tilde{A}| = 0.5$ and $\Delta\varphi_{BA} = 0.5$; the {normalized} magnitude of the complex 1-D longitudinal ***specific*** acoustic impedance $\left|\tilde{z}_{tot}^{\parallel}(x)\right|/z_o = \left|\tilde{z}_{tot}^{\parallel}(x)\right|/\rho_o c$ and its phase $\varphi_z(x) = \Delta\varphi_{p_{tot}-u_{tot}}(x) = \varphi_{p_{tot}}(x) - \varphi_{u_{tot}}(x)$ *vs.* kx are shown in the figure(s) below.



The ***time-averaged*** total/resultant complex 1-D longitudinal sound intensity associated with two counter-propagating monochromatic plane waves in “free air” is:

$$\left\langle \tilde{I}_{a_{tot}}^{\parallel}(x)\right\rangle_t \equiv \frac{1}{2}\tilde{p}_{tot}(x,t)\cdot\tilde{u}_{tot}^{\parallel *}(x,t) = \frac{1}{2}\tilde{A}\,\cancel{e^{i(\omega t-kx)}}\left[1+|\tilde{R}|e^{i(2kx+\Delta\varphi_{BA})}\right]\cdot\frac{\tilde{A}^*}{\rho_o c}\,\cancel{e^{-i(\omega t-kx)}}\left[1-|\tilde{R}|e^{-i(2kx+\Delta\varphi_{BA})}\right]$$

$$= \frac{1}{2}\frac{|\tilde{A}|^2}{\rho_o c}\left[1+|\tilde{R}|e^{i(2kx+\Delta\varphi_{BA})}\right]\cdot\left[1-|\tilde{R}|e^{-i(2kx+\Delta\varphi_{BA})}\right]$$

$$= \frac{1}{2}\frac{|\tilde{A}|^2}{\rho_o c}\left[1+|\tilde{R}|\left\{e^{i(2kx+\Delta\varphi_{BA})} - e^{-i(2kx+\Delta\varphi_{BA})}\right\} - |\tilde{R}|^2\right] \quad \text{and using: } z_o \equiv \rho_o c$$

$$= \frac{1}{2}\frac{|\tilde{A}|^2}{z_o}\left[1+2i|\tilde{R}|\sin(2kx+\Delta\varphi_{BA}) - |\tilde{R}|^2\right] = \frac{1}{2}\frac{|\tilde{A}|^2}{z_o}\left[\left\{1-|\tilde{R}|^2\right\}+2i|\tilde{R}|\sin(2kx+\Delta\varphi_{BA})\right]$$

Compare this expression to that for the complex 1-D longitudinal ***specific*** acoustic impedance:

$$\tilde{z}^{\parallel}_{a\,tot}(x) = z_o \frac{\left[\left\{1-\left|\tilde{R}\right|^2\right\} + 2i\left|\tilde{R}\right|\sin\left(2kx+\Delta\varphi_{BA}\right)\right]}{\left[\left\{1+\left|\tilde{R}\right|^2\right\} - 2\left|\tilde{R}\right|\cos\left(2kx+\Delta\varphi_{BA}\right)\right]}$$

Since $\left\langle \vec{\tilde{I}} \right\rangle_t \equiv \tfrac{1}{2}\tilde{p}\vec{\tilde{u}}^*$ and $\vec{\tilde{z}}_a \equiv \dfrac{\tilde{p}}{\vec{\tilde{u}}} = \dfrac{\tilde{p}}{\vec{\tilde{u}}}\dfrac{\bullet\vec{\tilde{u}}^*}{\bullet\vec{\tilde{u}}^*} = \dfrac{\tilde{p}\vec{\tilde{u}}^*}{\left|\vec{\tilde{u}}\right|^2} = \dfrac{2\left\langle \vec{\tilde{I}}_a \right\rangle_t}{\left|\vec{\tilde{u}}\right|^2}$, or $\left\langle \vec{\tilde{I}}_a(x) \right\rangle_t = \tfrac{1}{2}\left|\vec{\tilde{u}}(x,t)\right|^2 \vec{\tilde{z}}_a(x)$, then for the 1-D situation we have here with monochromatic counter-propagating traveling plane waves:

$$\left|\tilde{u}^{\parallel}_{tot}(x,t)\right|^2 = \frac{\left|\tilde{A}\right|^2}{z_o^2}\left[1-2\left|\tilde{R}\right|\cos\left(2kx+\Delta\varphi_{BA}\right)+\left|\tilde{R}\right|^2\right] = \frac{\left|\tilde{A}\right|^2}{z_o^2}\left[\left\{1+\left|\tilde{R}\right|^2\right\}-2\left|\tilde{R}\right|\cos\left(2kx+\Delta\varphi_{BA}\right)\right]$$

Thus we see that, indeed:

$$\frac{2\left\langle \tilde{I}^{\parallel}_{a_{tot}}(x)\right\rangle_t}{\left|\tilde{u}^{\parallel}_{tot}(x,t)\right|^2} = \frac{\cancel{2}\cdot\dfrac{1}{\cancel{2}}\dfrac{\cancel{\left|\tilde{A}\right|^2}}{\cancel{z_o}}\left[\left\{1-\left|\tilde{R}\right|^2\right\}+2i\left|\tilde{R}\right|\sin\left(2kx+\Delta\varphi_{BA}\right)\right]}{\dfrac{\cancel{\left|\tilde{A}\right|^2}}{z_o^{\cancel{2}}}\left[\left\{1+\left|\tilde{R}\right|^2\right\}-2\left|\tilde{R}\right|\cos\left(2kx+\Delta\varphi_{BA}\right)\right]} = z_o\frac{\left[\left\{1-\left|\tilde{R}\right|^2\right\}+2i\left|\tilde{R}\right|\sin\left(2kx+\Delta\varphi_{BA}\right)\right]}{\left[\left\{1+\left|\tilde{R}\right|^2\right\}-2\left|\tilde{R}\right|\cos\left(2kx+\Delta\varphi_{BA}\right)\right]} = \tilde{z}^{\parallel}_{a\,tot}(x)$$

Note again that the above expression for the ***time-averaged*** total/resultant complex 1-D longitudinal sound intensity $\left\langle \tilde{I}^{\parallel}_{a_{tot}}(x)\right\rangle_t$, like that for the complex 1-D longitudinal specific acoustic impedance $\tilde{z}^{\parallel}_{a\,tot}(x)$ has ***no*** explicit time dependence.

The active and reactive components of the ***instantaneous*** total/resultant complex 1-D longitudinal sound intensity are respectively defined as:

$$\tilde{I}^{\parallel inst}_{a_{tot}\,r}(x,t) \equiv p_{tot\,r}(x,t)\cdot u^{\parallel}_{tot\,r}(x,t) \quad \text{and} \quad \tilde{I}^{\parallel inst}_{a_{tot}\,i}(x,t) \equiv p_{tot\,r}(x,t)\cdot u^{\parallel}_{tot\,i}(x,t)$$

But:

$$\begin{aligned}
\tilde{p}_{tot}(x,t) &= \tilde{A}e^{i(\omega t-kx)}\left[1 + \left|\tilde{R}\right|e^{i(2kx+\Delta\varphi_{BA})}\right] = \left|\tilde{A}\right|e^{i\varphi_A}\cdot e^{i(\omega t-kx)}\left[1 + \left|\tilde{R}\right|e^{i(2kx+\Delta\varphi_{BA})}\right] \\
&= \left|\tilde{A}\right|\left[e^{i(\omega t-kx+\varphi_A)}+\left|\tilde{R}\right|e^{i(\omega t-kx+\varphi_A)}e^{i(2kx+\Delta\varphi_{BA})}\right] = \left|\tilde{A}\right|\left[e^{i(\omega t-kx+\varphi_A)}+\left|\tilde{R}\right|e^{i(\omega t+kx+\varphi_B)}\right] \\
&= \left|\tilde{A}\right|\left[\left\{\cos\left(\omega t-kx+\varphi_A\right)+\left|\tilde{R}\right|\cos\left(\omega t+kx+\varphi_B\right)\right\}+i\left\{\sin\left(\omega t-kx+\varphi_A\right)+\left|\tilde{R}\right|\sin\left(\omega t+kx+\varphi_B\right)\right\}\right]
\end{aligned}$$

Thus:

$$p_{tot\,r}(x,t) = |\tilde{A}|\left[\cos(\omega t - kx + \varphi_A) + |\tilde{R}|\cos(\omega t + kx + \varphi_B)\right]$$

$$p_{tot\,i}(x,t) = |\tilde{A}|\left[\sin(\omega t - kx + \varphi_A) \ + |\tilde{R}|\sin(\omega t + kx + \varphi_B)\right]$$

{ So: $\varphi_p(x,t) = \tan^{-1}\left(\dfrac{p_{tot\,i}(x,t)}{p_{tot\,r}(x,t)}\right) = \tan^{-1}\left(\dfrac{\sin(\omega t - kx + \varphi_A) + |\tilde{R}|\sin(\omega t + kx + \varphi_B)}{\cos(\omega t - kx + \varphi_A) + |\tilde{R}|\cos(\omega t + kx + \varphi_B)}\right)$ } and:

$$\begin{aligned}\tilde{u}^{\|}_{tot}(x,t) &= \frac{\tilde{A}}{\rho_o c} e^{i(\omega t - kx)}\left[1 - |\tilde{R}|e^{i(2kx + \Delta\varphi_{BA})}\right] = \frac{|\tilde{A}|}{\rho_o c} e^{i\varphi_A} \cdot e^{i(\omega t - kx)}\left[1 - |\tilde{R}|e^{i(2kx + \Delta\varphi_{BA})}\right] \\ &= \frac{|\tilde{A}|}{\rho_o c} e^{i(\omega t - kx + \varphi_A)}\left[1 - |\tilde{R}|e^{i(2kx + \Delta\varphi_{BA})}\right] = \frac{|\tilde{A}|}{\rho_o c}\left[e^{i(\omega t - kx + \varphi_A)} - |\tilde{R}|e^{i(\omega t + kx + \varphi_B)}\right] \\ &= \frac{|\tilde{A}|}{\rho_o c}\left[\left\{\cos(\omega t - kx + \varphi_A) - |\tilde{R}|\cos(\omega t + kx + \varphi_B)\right\} + i\left\{\sin(\omega t - kx + \varphi_A) - |\tilde{R}|\sin(\omega t + kx + \varphi_B)\right\}\right]\end{aligned}$$

Thus:

$$u^{\|}_{tot\,r}(x,t) = \frac{|\tilde{A}|}{\rho_o c}\left[\cos(\omega t - kx + \varphi_A) - |\tilde{R}|\cos(\omega t + kx + \varphi_B)\right]$$

$$u^{\|}_{tot\,i}(x,t) = \frac{|\tilde{A}|}{\rho_o c}\left[\sin(\omega t - kx + \varphi_A) - |\tilde{R}|\sin(\omega t + kx + \varphi_B)\right]$$

{ So: $\varphi_u(x,t) = \tan^{-1}\left(\dfrac{u^{\|}_{tot\,i}(x,t)}{u^{\|}_{tot\,r}(x,t)}\right) = \tan^{-1}\left(\dfrac{\sin(\omega t - kx + \varphi_A) - |\tilde{R}|\sin(\omega t + kx + \varphi_B)}{\cos(\omega t - kx + \varphi_A) - |\tilde{R}|\cos(\omega t + kx + \varphi_B)}\right)$ }

Hence:

$$\begin{aligned}\tilde{I}^{\|\,inst}_{a_{tot}\,r}(x,t) &\equiv p_{tot\,r}(x,t)\cdot u^{\|}_{tot\,r}(x,t) \\ &= \frac{|\tilde{A}|^2}{\rho_o c}\left[\cos(\omega t - kx + \varphi_A) + |\tilde{R}|\cos(\omega t + kx + \varphi_B)\right]\cdot\left[\cos(\omega t - kx + \varphi_A) - |\tilde{R}|\cos(\omega t + kx + \varphi_B)\right] \\ &= \frac{|\tilde{A}|^2}{z_o}\left[\cos^2(\omega t - kx + \varphi_A) - |\tilde{R}|^2\cos^2(\omega t + kx + \varphi_B)\right] \quad \text{using: } z_o = \rho_o c\end{aligned}$$

and:

$$\begin{aligned}\tilde{I}^{\|\,inst}_{a_{tot}\,i}(x,t) &\equiv p_{tot\,r}(x,t)\cdot u^{\|}_{tot\,i}(x,t) \\ &= \frac{|\tilde{A}|^2}{\rho_o c}\left[\cos(\omega t - kx + \varphi_A) + |\tilde{R}|\cos(\omega t + kx + \varphi_B)\right]\cdot\left[\sin(\omega t - kx + \varphi_A) - |\tilde{R}|\sin(\omega t + kx + \varphi_B)\right] \\ &= \frac{|\tilde{A}|^2}{z_o}\left[\begin{array}{l}\cos(\omega t - kx + \varphi_A)\sin(\omega t - kx + \varphi_A) - |\tilde{R}|^2\sin(\omega t + kx + \varphi_B)\cos(\omega t + kx + \varphi_B) \\ -|\tilde{R}|\cos(\omega t - kx + \varphi_A)\sin(\omega t + kx + \varphi_B) + |\tilde{R}|\sin(\omega t - kx + \varphi_A)\cos(\omega t + kx + \varphi_B)\end{array}\right]\end{aligned}$$

For an observer/listener's position at $x = 0$ and setting $\varphi_A = 0$, $\Delta\varphi_{BA} \equiv \varphi_B - \varphi_A = \varphi_B - 0 = \varphi_B$ and $|\tilde{R}| = 1$ (i.e. a "pure" standing wave), the complex pressure, 1-D longitudinal particle velocity, ***specific*** acoustic impedance, time-averaged sound intensity and active and reactive components of the instantaneous sound intensity simplify to:

$$\tilde{p}_{tot}(x=0,t) = \tilde{A}e^{i\omega t}\left[1+e^{i\Delta\varphi_{BA}}\right] = \tilde{A}e^{i\omega t}\left[\{1+\cos\Delta\varphi_{BA}\}+i\sin\Delta\varphi_{BA}\right]$$

$$\tilde{u}_{tot}^{\parallel}(x=0,t) = \frac{\tilde{A}}{z_o}e^{i\omega t}\left[1-e^{i\Delta\varphi_{BA}}\right] = \frac{\tilde{A}}{z_o}e^{i\omega t}\left[\{1-\cos\Delta\varphi_{BA}\}-i\sin\Delta\varphi_{BA}\right]$$

$$\tilde{z}_{a\,tot}^{\parallel}(x=0) = z_o\frac{i\sin\Delta\varphi_{BA}}{[1-\cos\Delta\varphi_{BA}]}$$

$$\left\langle \tilde{I}_{a_{tot}}^{\parallel}(x)\right\rangle_t \equiv \frac{1}{2}\tilde{p}_{tot}(x=0,t)\cdot\tilde{u}_{tot}^{\parallel *}(x=0,t) = i\frac{|\tilde{A}|^2}{z_o}\sin\Delta\varphi_{BA}$$

$$\tilde{I}_{a_{tot}\,r}^{\parallel inst}(x=0,t) \equiv p_{tot\,r}(x=0,t)\cdot u_{tot\,r}^{\parallel}(x=0,t) = \frac{|\tilde{A}|^2}{z_o}\left[\cos^2\omega t - \cos^2(\omega t+\Delta\varphi_{AB})\right]$$

$$= \frac{|\tilde{A}|^2}{z_o}\left[\cos^2\omega t - \{\cos\omega t\cos\Delta\varphi_{AB} - \sin\omega t\sin\Delta\varphi_{AB}\}^2\right]$$

$$= \frac{|\tilde{A}|^2}{z_o}\left[\cos^2\omega t - \cos^2\omega t\cos^2\Delta\varphi_{AB} + 2\cos\omega t\sin\omega t\cos\Delta\varphi_{AB}\sin\Delta\varphi_{AB} - \sin^2\omega t\sin^2\Delta\varphi_{AB}\right]$$

$$= \frac{|\tilde{A}|^2}{z_o}\left[\cos^2\omega t\left(1-\cos^2\Delta\varphi_{AB}\right) + 2\cos\omega t\sin\omega t\cos\Delta\varphi_{AB}\sin\Delta\varphi_{AB} - \sin^2\omega t\sin^2\Delta\varphi_{AB}\right]$$

$$= \frac{|\tilde{A}|^2}{z_o}\left[\cancel{\cos^2\omega t\sin^2\Delta\varphi_{AB}} + 2\cos\omega t\sin\omega t\cos\Delta\varphi_{AB}\sin\Delta\varphi_{AB} - \cancel{\sin^2\omega t\sin^2\Delta\varphi_{AB}}\right]$$

$$= 2\frac{|\tilde{A}|^2}{z_o}\left[\cos\omega t\sin\omega t\cos\Delta\varphi_{AB}\sin\Delta\varphi_{AB}\right]$$

and:

$$\tilde{I}^{\parallel inst}_{a_{tot}\,i}(x,t) \equiv p_{tot\,r}(x,t)\cdot u^{\parallel}_{tot\,i}(x,t)$$

$$= \frac{|\tilde{A}|^2}{z_o}\left[\begin{array}{l}\cos\omega t\sin\omega t - \sin(\omega t + \Delta\varphi_{BA})\cos(\omega t + \Delta\varphi_{BA}) \\ -\cos\omega t\sin(\omega t + \Delta\varphi_{BA}) + \sin\omega t\cos(\omega t + \Delta\varphi_{BA})\end{array}\right]$$

$$= \frac{|\tilde{A}|^2}{z_o}\left[\begin{array}{l}\cos\omega t\sin\omega t - \{\sin\omega t\cos\Delta\varphi_{BA} + \cos\omega t\sin\Delta\varphi_{BA}\}\cdot\{\cos\omega t\cos\Delta\varphi_{BA} - \sin\omega t\sin\Delta\varphi_{BA}\} \\ -\cos\omega t\cdot\{\sin\omega t\cos\Delta\varphi_{BA} + \cos\omega t\sin\Delta\varphi_{BA}\} + \sin\omega t\cdot\{\cos\omega t\cos\Delta\varphi_{BA} - \sin\omega t\sin\Delta\varphi_{BA}\}\end{array}\right]$$

$$= \frac{|\tilde{A}|^2}{z_o}\left[\begin{array}{l}\cos\omega t\sin\omega t - \cos\omega t\sin\omega t\left(\cos^2\Delta\varphi_{BA} - \sin^2\Delta\varphi_{BA}\right) - \left(\cos^2\omega t - \sin^2\omega t\right)\cos\Delta\varphi_{BA}\sin\Delta\varphi_{BA} \\ \cancel{-\cos\omega t\sin\omega t\cos\Delta\varphi_{BA}} - \cos^2\omega t\sin\Delta\varphi_{BA} + \cancel{\cos\omega t\sin\omega t\cos\Delta\varphi_{BA}} - \sin^2\omega t\sin\Delta\varphi_{BA}\end{array}\right]$$

$$= \frac{|\tilde{A}|^2}{z_o}\left[\begin{array}{l}\cos\omega t\sin\omega t\left[1 - \left(\cos^2\Delta\varphi_{BA} - \sin^2\Delta\varphi_{BA}\right)\right] - \left(\cos^2\omega t - \sin^2\omega t\right)\cos\Delta\varphi_{BA}\sin\Delta\varphi_{BA} \\ -\left(\cos^2\omega t + \sin^2\omega t\right)\sin\Delta\varphi_{BA}\end{array}\right]$$

$$= \frac{|\tilde{A}|^2}{z_o}\left[-2\cos\omega t\sin\omega t\sin^2\Delta\varphi_{BA} - \left(\cos^2\omega t - \sin^2\omega t\right)\cos\Delta\varphi_{BA}\sin\Delta\varphi_{BA} - \sin\Delta\varphi_{BA}\right]$$

If we take the time-averages of the active and reactive components of the above instantaneous 1-D sound intensities, using the fact that $\langle\cos^2\omega t\rangle_t = \langle\sin^2\omega t\rangle_t = \frac{1}{2}$ and $\langle\cos\omega t\sin\omega t\rangle_t = 0$ we see that:

$$\left\langle\tilde{I}^{\parallel}_{a_{tot}}(x)\right\rangle_t \equiv \frac{1}{2}\tilde{p}_{tot}(x=0,t)\cdot\tilde{u}^{\parallel *}_{tot}(x=0,t) = 0 + i\frac{|\tilde{A}|^2}{z_o}\sin\Delta\varphi_{BA} = i\frac{|\tilde{A}|^2}{z_o}\sin\Delta\varphi_{BA}$$

$$\left\langle\tilde{I}^{\parallel inst}_{a_{tot}\,r}(x=0,t)\right\rangle_t \equiv \left\langle p_{tot\,r}(x=0,t)\cdot u^{\parallel}_{tot\,r}(x=0,t)\right\rangle_t = 2\frac{|\tilde{A}|^2}{z_o}\left[\cancel{\langle\cos\omega t\sin\omega t\rangle}\cos\Delta\varphi_{BA}\sin\Delta\varphi_{BA}\right] = 0 \quad \surd$$

$$\left\langle\tilde{I}^{\parallel inst}_{a_{tot}\,i}(x=0,t)\right\rangle_t \equiv \left\langle p_{tot\,r}(x,t)\cdot u^{\parallel}_{tot\,i}(x,t)\right\rangle_t$$

$$= \frac{|\tilde{A}|^2}{z_o}\left[-2\cancel{\langle\cos\omega t\sin\omega t\rangle}\sin^2\Delta\varphi_{BA} - \left(\langle\cos^2\omega t\rangle - \langle\sin^2\omega t\rangle\right)\cos\Delta\varphi_{BA}\sin\Delta\varphi_{BA} - \sin\Delta\varphi_{BA}\right]$$

$$= \frac{|\tilde{A}|^2}{z_o}\left[-\cancel{\left(\frac{1}{2} - \frac{1}{2}\right)}\cos\Delta\varphi_{BA}\sin\Delta\varphi_{BA} - \sin\Delta\varphi_{BA}\right] = -\frac{|\tilde{A}|^2}{z_o}\sin\Delta\varphi_{BA} \quad \surd$$

If we Taylor series-expand $\sin\Delta\varphi_{BA}$ and $\cos\Delta\varphi_{BA}$ e.g. about $\Delta\varphi_{BA} \approx 0$, i.e.

$$\sin\Delta\varphi_{BA} = \Delta\varphi_{BA} - (\Delta\varphi_{BA})^3/3! + (\Delta\varphi_{BA})^5/5! - (\Delta\varphi_{BA})^7/7! + \ldots \sim \Delta\varphi_{BA}$$

and:

$$\cos\Delta\varphi_{BA} = 1 - (\Delta\varphi_{BA})^2/2! + (\Delta\varphi_{BA})^4/4! - (\Delta\varphi_{BA})^6/6! + \ldots. \quad \sim 1 - (\Delta\varphi_{BA})^2/2$$

Then:

$$\tilde{p}_{tot}\left(x=0,t\right)=\tilde{A}e^{i\omega t}\left[\left\{1+\cos\Delta\varphi_{BA}\right\}+i\sin\Delta\varphi_{BA}\right]\simeq\tilde{A}e^{i\omega t}\left[2\left\{1+\tfrac{1}{4}\left(\Delta\varphi_{BA}\right)^2\right\}+i\Delta\varphi_{BA}\right]$$

$$\tilde{u}_{tot}^{\parallel}\left(x=0,t\right)=\frac{\tilde{A}}{z_o}e^{i\omega t}\left[\left\{1-\cos\Delta\varphi_{BA}\right\}-i\sin\Delta\varphi_{BA}\right]\simeq\frac{\tilde{A}}{z_o}e^{i\omega t}\left[\tfrac{1}{2}\left(\Delta\varphi_{BA}\right)^2-i\Delta\varphi_{BA}\right]\approx -i\frac{\tilde{A}}{z_o}e^{i\omega t}\Delta\varphi_{BA}$$

$$\tilde{z}_{a\,tot}^{\parallel}\left(x=0\right)=z_o\frac{i\sin\Delta\varphi_{BA}}{\left[1-\cos\Delta\varphi_{BA}\right]}\simeq z_o\frac{i\Delta\varphi_{BA}}{\left[1-1+\tfrac{1}{2}\left(\Delta\varphi_{BA}\right)^2\right]}=z_o\frac{2i\,\cancel{\Delta\varphi_{BA}}}{\left(\Delta\varphi_{BA}\right)^{\cancel{2}}}=\frac{2z_o i}{\Delta\varphi_{BA}}$$

Then in the limit $\Delta\varphi_{BA}\to 0$:

$$\lim_{\Delta\varphi_{BA}\to 0}\left\{\tilde{p}_{tot}\left(x=0,t\right)\right\}\simeq\tilde{A}e^{i\omega t}\left[2\left\{1+\tfrac{1}{4}\left(\Delta\varphi_{BA}\right)^2\right\}+i\Delta\varphi_{BA}\right]\Bigg|_{\Delta\varphi_{BA}\to 0}\approx\tilde{A}e^{i\omega t}\left(2+i\Delta\varphi_{BA}\right)\Big|_{\Delta\varphi_{BA}\to 0}$$

$$\lim_{\Delta\varphi_{BA}\to 0}\left\{\tilde{u}_{tot}^{\parallel}\left(x=0,t\right)\right\}\simeq\frac{\tilde{A}}{z_o}e^{i\omega t}\left[\tfrac{1}{2}\left(\Delta\varphi_{BA}\right)^2-i\Delta\varphi_{BA}\right]\Bigg|_{\Delta\varphi_{BA}\to 0}\approx -i\Delta\varphi_{BA}\cdot\frac{\tilde{A}}{z_o}e^{i\omega t}\Bigg|_{\Delta\varphi_{BA}\to 0}$$

$$\lim_{\Delta\varphi_{BA}\to 0}\left\{\tilde{z}_{a\,tot}^{\parallel}\left(x=0\right)\right\}=\lim_{\Delta\varphi_{BA}\to 0}\left\{\frac{\tilde{p}_{tot}\left(x=0,t\right)}{\tilde{u}_{tot}^{\parallel}\left(x=0,t\right)}\right\}\approx\left(\frac{\cancel{\tilde{A}}\,\cancel{e^{i\omega t}}\left(2+i\Delta\varphi_{BA}\right)}{-i\Delta\varphi_{BA}\cdot\left(\cancel{\tilde{A}}/z_o\right)\cancel{e^{i\omega t}}}\right)\Bigg|_{\Delta\varphi_{BA}\to 0}$$

$$\approx\frac{2z_o i}{\Delta\varphi_{BA}}\Bigg|_{\Delta\varphi_{BA}\to 0}=\infty\cdot i\cdot sign\left\{\Delta\varphi_{BA}\right\}$$

Thus, we see that when $\Delta\varphi_{BA}=0,\pm 2\pi,\pm 4\pi,\pm 6\pi,\ldots=\pm n_{even}\pi$ that $\tilde{z}_{a\,tot}^{\parallel}\left(x=0\right)=\infty\cdot i\cdot sign\left\{\Delta\varphi_{BA}\right\}$ and $\tilde{I}_{a_{tot}}^{\parallel\,inst}\left(x=0,t\right)=0$.

However, when $\Delta\varphi_{BA}=\pm 1\pi,\pm 3\pi,\pm 5\pi,\ldots=\pm n_{odd}\pi$ we see that <u>*both*</u> $\tilde{z}_{a\,tot}^{\parallel}\left(x=0\right)$ and $\tilde{I}_{a_{tot}}^{\parallel\,inst}\left(x=0,t\right)=0$.

When $\Delta\varphi_{BA}=\pm\pi/2,\pm 3\pi/2,\pm 5\pi/2,\ldots=\pm m_{odd}\pi/2$ then: $\tilde{z}_{a\,tot}^{\parallel}\left(x=0\right)=\pm i z_o$ and because $\cos\left(\omega t\pm m_{odd}\pi/2\right)=\cos\omega t\,\cancel{\cos\left(m_{odd}\pi/2\right)}\mp\sin\omega t\sin\left(m_{odd}\pi/2\right)=\mp\sin\omega t$ then:

$$\tilde{I}_{a_{tot}\,r}^{\parallel\,inst}\left(x=0,t\right)=\frac{\left|\tilde{A}\right|^2}{z_o}\left[\cos^2\omega t-\sin^2\omega t\right]=\frac{\left|\tilde{A}\right|^2}{z_o}\left[1-2\sin^2\omega t\right]$$

Notice that for $\left|\tilde{R}\right|=1$, we could have learned about the above two null results directly from:

$$\left\langle\tilde{I}_{a_{tot}}^{\parallel}\left(x\right)\right\rangle_t\equiv\frac{1}{2}\tilde{p}_{tot}\left(x,t\right)\cdot\tilde{u}_{tot}^{\parallel *}\left(x,t\right)=\frac{1}{2}\frac{\left|\tilde{A}\right|^2}{z_o}\left[\left\{1-\left|\tilde{R}\right|^2\right\}+2i\left|\tilde{R}\right|\sin\left(2kx+\Delta\varphi_{BA}\right)\right]=\left\langle\tilde{I}_{a_{tot}\,r}^{\parallel}\left(x\right)\right\rangle_t+i\left\langle\tilde{I}_{a_{tot}\,i}^{\parallel}\left(x\right)\right\rangle_t$$

For an observer/listener's position at x = 0 and setting $\varphi_A = 0$, $\Delta\varphi_{BA} \equiv \varphi_B - \varphi_A = \varphi_B - 0 = \varphi_B$ and $|\tilde{R}| = 1$ (i.e. a "pure" standing wave), this simplifies to:

$$\left\langle \tilde{I}^{\parallel}_{a_{tot}}(x=0) \right\rangle_t \equiv \frac{1}{2}\tilde{p}_{tot}(x=0,t)\cdot\tilde{u}^{\parallel *}_{tot}(x=0,t) = i\frac{|\tilde{A}|^2}{z_o}\sin\Delta\varphi_{BA} \text{ (n.b. purely imaginary quantity!)}$$

We see again that when $\Delta\varphi_{AB} = 0, \pm 1\pi, \pm 2\pi, \pm 3\pi, ... = \pm n\pi$ that $\tilde{I}^{\parallel}_{a_{tot}}(x=0,t) = 0$!!! Similarly, we see that it also has a purely ***imaginary*** ***extremum*** amplitude of $\pm|\tilde{A}|^2 / \rho_o c = \pm|\tilde{A}|^2 / z_o$ when $\Delta\varphi_{AB} = \pm\pi/2, \pm 3\pi/2, \pm 5\pi/2, ... = \pm m_{odd}\pi/2$.

The ***instantaneous*** potential, kinetic and total energy densities associated with counter-propagating 1-D monochromatic traveling plane waves are:

$$w^{inst}_{potl}(x,t) \equiv \frac{1}{2}\frac{1}{\rho_o c^2}p^2_{tot\,r}(x,t) \qquad = \frac{1}{2}\frac{|\tilde{A}|^2}{\rho_o c^2}\left[\cos(\omega t - kx + \varphi_A) + |\tilde{R}|\cos(\omega t + kx + \varphi_B)\right]^2$$

$$= \frac{1}{2}\frac{|\tilde{A}|^2}{\rho_o c^2}\left[\cos^2(\omega t - kx + \varphi_A) + 2|\tilde{R}|\cos(\omega t - kx + \varphi_A)\cos(\omega t + kx + \varphi_B) + |\tilde{R}|^2\cos^2(\omega t + kx + \varphi_B)\right]$$

$$w^{inst}_{kin}(x,t) \equiv \frac{1}{2}\rho_o u^{\parallel}_{tot\,r}(x,t)\bullet u^{\parallel}_r(x,t) = \frac{1}{2}\frac{|\tilde{A}|^2}{\rho_o c^2}\left[\cos(\omega t - kx + \varphi_A) - |\tilde{R}|\cos(\omega t + kx + \varphi_B)\right]^2$$

$$= \frac{1}{2}\frac{|\tilde{A}|^2}{\rho_o c^2}\left[\cos^2(\omega t - kx + \varphi_A) - 2|\tilde{R}|\cos(\omega t - kx + \varphi_A)\cos(\omega t + kx + \varphi_B) + |\tilde{R}|^2\cos^2(\omega t + kx + \varphi_B)\right]$$

$$w^{inst}_{tot}(x,t) \equiv w^{inst}_{potl}(x,t) + w^{inst}_{kin}(x,t) = \frac{|\tilde{A}|^2}{\rho_o c^2}\left[\cos^2(\omega t - kx + \varphi_A) + |\tilde{R}|^2\cos^2(\omega t + kx + \varphi_B)\right]$$

Again, for an observer's position at x = 0 and setting $\varphi_A = 0$, $\Delta\varphi_{BA} \equiv \varphi_B - \varphi_A = \varphi_B - 0 = \varphi_B$ and $|\tilde{R}| = 1$ (i.e. a "pure" standing wave), these quantities simplify to:

$$w^{inst}_{potl}(x=0,t) = \frac{1}{2}\frac{|\tilde{A}|^2}{\rho_o c^2}\left[\cos\omega t + \cos(\omega t + \Delta\varphi_{BA})\right]^2$$

$$w^{inst}_{kin}(x=0,t) = \frac{1}{2}\frac{|\tilde{A}|^2}{\rho_o c^2}\left[\cos\omega t - \cos(\omega t + \Delta\varphi_{BA})\right]^2$$

$$w^{inst}_{tot}(x,t) \equiv w^{inst}_{potl}(x,t) + w^{inst}_{kin}(x,t) = \frac{|\tilde{A}|^2}{\rho_o c^2}\left[\cos^2\omega t + |\tilde{R}|^2\cos^2(\omega t + \Delta\varphi_{BA})\right]$$

We see that when: $\Delta\varphi_{BA} = 0, \pm 2\pi, \pm 4\pi, \pm 6\pi, ... = \pm n_{even}\pi$ that since:
$\cos(\theta \pm n_{even}\pi) = (\cos\theta \cdot \cos n_{even}\pi) \mp (\sin\theta \cdot \cancel{\sin n_{even}\pi}) = \cos\theta$, hence:

$$w_{potl}^{inst}(x=0,t) = \frac{1}{2}\frac{|\tilde{A}|^2}{\rho_o c^2}\left[\cos\omega t + \cos(\omega t \pm n_{even}\pi)\right]^2 = \frac{1}{2}\frac{4|\tilde{A}|^2}{\rho_o c^2}\cos^2\omega t = \frac{2|\tilde{A}|^2}{\rho_o c^2}\cos^2\omega t = \frac{2|\tilde{A}|^2}{z_o c}\cos^2\omega t$$

$$w_{kin}^{inst}(x=0,t) = \frac{1}{2}\frac{|\tilde{A}|^2}{\rho_o c^2}\left[\cos\omega t - \cos(\omega t + \Delta\varphi_{BA})\right]^2 = 0$$

$$w_{tot}^{inst}(x=0,t) \equiv w_{potl}^{inst}(x=0,t) + w_{kin}^{inst}(x=0,t) = \frac{2|\tilde{A}|^2}{\rho_o c^2}\cos^2\omega t = \frac{2|\tilde{A}|^2}{z_o c}\cos^2\omega t$$

We also see that when $\Delta\varphi_{BA} = \pm 1\pi, \pm 3\pi, \pm 5\pi, ... = \pm n_{odd}\pi$ that since:
$\cos(\theta \pm n_{odd}\pi) = (\cos\theta \cdot \cos n_{odd}\pi) \mp (\sin\theta \cdot \cancel{\sin n_{odd}\pi}) = -\cos\theta$, hence:

$$w_{potl}^{inst}(x=0,t) = \frac{1}{2}\frac{|\tilde{A}|^2}{\rho_o c^2}\left[\cos\omega t + \cos(\omega t \pm n_{odd}\pi)\right]^2 = 0$$

$$w_{kin}^{inst}(x=0,t) = \frac{1}{2}\frac{|\tilde{A}|^2}{\rho_o c^2}\left[\cos\omega t - \cos(\omega t + \Delta\varphi_{BA})\right]^2 = \frac{1}{2}\frac{4|\tilde{A}|^2}{\rho_o c^2}\cos^2\omega t = \frac{2|\tilde{A}|^2}{\rho_o c^2}\cos^2\omega t$$

$$w_{tot}^{inst}(x=0,t) \equiv w_{potl}^{inst}(x=0,t) + w_{kin}^{inst}(x=0,t) = \frac{2|\tilde{A}|^2}{\rho_o c^2}\cos^2\omega t = \frac{2|\tilde{A}|^2}{z_o c}\cos^2\omega t$$

The ***time-averaged*** potential, kinetic and total energy densities associated with counter-propagating 1-D monochromatic traveling plane waves are defined as:

$$\langle w_{potl}(x)\rangle_t \equiv \frac{1}{4}\frac{1}{\rho_o c^2}|\tilde{p}_{tot}(x,t)|^2 = \frac{1}{4}\frac{|\tilde{A}|^2}{\rho_o c^2}\left[1 + |\tilde{R}|^2 + 2|\tilde{R}|\cos(2kx + \Delta\varphi_{BA})\right]$$

$$\langle w_{kin}(x)\rangle_t \equiv \frac{1}{4}\rho_o|\tilde{u}_{tot}^{\parallel}(x,t)|^2 \quad = \frac{1}{4}\frac{|\tilde{A}|^2}{\rho_o c^2}\left[1 + |\tilde{R}|^2 - 2|\tilde{R}|\cos(2kx + \Delta\varphi_{BA})\right]$$

$$\langle w_{tot}(x)\rangle_t \equiv \langle w_{potl}(x)\rangle_t + \langle w_{kin}(x)\rangle_t = \frac{1}{2}\frac{|\tilde{A}|^2}{\rho_o c^2}\left[1 + |\tilde{R}|^2\right] = \frac{1}{2}\frac{|\tilde{A}|^2}{z_o c}\left[1 + |\tilde{R}|^2\right]$$

Note that here, the ratio of the ***time-averaged*** potential energy density to the ***time-averaged*** kinetic energy density is ***not*** equal to unity for counter-propagating monochromatic plane waves:

$$\frac{\langle w_{potl}(x)\rangle_t}{\langle w_{kin}(x)\rangle_t} = \frac{\frac{1}{4}\frac{|\tilde{p}(x,t)|^2}{\rho_o c^2}}{\frac{1}{4}\rho_o|\vec{\tilde{u}}(x,t)|^2} = \frac{\left[1 + |\tilde{R}|^2 + 2|\tilde{R}|\cos(2kx + \Delta\varphi_{BA})\right]}{\left[1 + |\tilde{R}|^2 - 2|\tilde{R}|\cos(2kx + \Delta\varphi_{BA})\right]} \neq 1$$

Again, for an observer's position at $x = 0$ and setting $\varphi_A = 0$, $\Delta\varphi_{BA} \equiv \varphi_B - \varphi_A = \varphi_B - 0 = \varphi_B$ and $|\tilde{R}| = 1$ (i.e. a "pure" standing wave), these quantities simplify to:

$$\left\langle w_{potl}(x=0)\right\rangle_t \equiv \frac{1}{4}\frac{1}{\rho_o c^2}\left|\tilde{p}_{tot}(x=0,t)\right|^2 = \frac{1}{2}\frac{|\tilde{A}|^2}{\rho_o c^2}\left[1+\cos\Delta\varphi_{BA}\right]$$

$$\left\langle w_{kin}(x=0)\right\rangle_t \equiv \frac{1}{4}\rho_o\left|\tilde{u}^{\parallel}_{tot}(x=0,t)\right|^2 \quad = \frac{1}{2}\frac{|\tilde{A}|^2}{\rho_o c^2}\left[1-\cos\Delta\varphi_{BA}\right]$$

$$\left\langle w_{tot}(x=0)\right\rangle_t \equiv \left\langle w_{potl}(x=0)\right\rangle_t + \left\langle w_{kin}(x=0)\right\rangle_t$$

$$= \frac{1}{2}\frac{|\tilde{A}|^2}{\rho_o c^2}\left[1+\cancel{\cos\Delta\varphi_{BA}}\right] + \frac{1}{2}\frac{|\tilde{A}|^2}{\rho_o c^2}\left[1-\cancel{\cos\Delta\varphi_{BA}}\right] = \frac{|\tilde{A}|^2}{\rho_o c^2} = \frac{|\tilde{A}|^2}{z_o c}$$

We see that when: $\Delta\varphi_{BA} = 0, \pm 2\pi, \pm 4\pi, \pm 6\pi, ... = \pm n_{even}\pi$ that:

$$\left\langle w_{potl}(x=0)\right\rangle_t \equiv \frac{1}{4}\frac{1}{\rho_o c^2}\left|\tilde{p}_{tot}(x=0,t)\right|^2 = \frac{1}{2}\frac{|\tilde{A}|^2}{\rho_o c^2}\left[1+\cos(\pm n_{even}\pi)\right] = \frac{|\tilde{A}|^2}{\rho_o c^2}$$

$$\left\langle w_{kin}(x=0)\right\rangle_t \equiv \frac{1}{4}\rho_o\left|\tilde{u}^{\parallel}_{tot}(x=0,t)\right|^2 \quad = \frac{1}{2}\frac{|\tilde{A}|^2}{\rho_o c^2}\left[1-\cos(\pm n_{even}\pi)\right] = 0$$

$$\left\langle w_{tot}(x=0)\right\rangle_t \equiv \left\langle w_{potl}(x=0)\right\rangle_t + \left\langle w_{kin}(x=0)\right\rangle_t = \frac{|\tilde{A}|^2}{\rho_o c^2} = \frac{|\tilde{A}|^2}{z_o c}$$

We also see that when $\Delta\varphi_{AB} = \pm 1\pi, \pm 3\pi, \pm 5\pi, ... = \pm n_{odd}\pi$ that:

$$\left\langle w_{potl}(x=0)\right\rangle_t \equiv \frac{1}{4}\frac{1}{\rho_o c^2}\left|\tilde{p}_{tot}(x=0,t)\right|^2 = \frac{1}{2}\frac{|\tilde{A}|^2}{\rho_o c^2}\left[1+\cos(\pm n_{odd}\pi)\right] = 0$$

$$\left\langle w_{kin}(x=0)\right\rangle_t \equiv \frac{1}{4}\rho_o\left|\tilde{u}^{\parallel}_{tot}(x=0,t)\right|^2 \quad = \frac{1}{2}\frac{|\tilde{A}|^2}{\rho_o c^2}\left[1-\cos(\pm n_{odd}\pi)\right] = \frac{|\tilde{A}|^2}{\rho_o c^2}$$

$$\left\langle w_{tot}(x=0)\right\rangle_t \equiv \left\langle w_{potl}(x=0)\right\rangle_t + \left\langle w_{kin}(x=0)\right\rangle_t = \frac{|\tilde{A}|^2}{\rho_o c^2}$$

Physically, the <u>real</u> part of the <u>***time-averaged***</u> complex sound intensity represents a {net/time-averaged} flux/flow of energy crossing unit area per unit time (n.b. its *SI* units are *Watts*/m^2).

When the counter-propagating monochromatic plane waves are <u>*either*</u> precisely in-phase with each other ($\Delta\varphi_{BA} = 0, \pm 2\pi, \pm 4\pi, \pm 6\pi, ... = \pm n_{even}\pi$) or precisely out-of-phase with each other ($\Delta\varphi_{BA} = 0, \pm 2\pi, \pm 4\pi, \pm 6\pi, ... = \pm n_{even}\pi$) then for a <u>*pure*</u> standing wave ($|\tilde{R}| \equiv |\tilde{B}| / |\tilde{A}| = 1$) e.g. at the position $x = 0$, the <u>real</u> part of the instantaneous 1-D longitudinal sound intensity is:

$$\tilde{I}^{\parallel inst}_{a_{tot}\, r}(x=0,t) \equiv p_{tot\, r}(x=0,t)\cdot u^{\parallel}_{tot\, r}(x=0,t) = \frac{|\tilde{A}|^2}{z_o}\left[\cos^2\omega t - \cos^2(\omega t + \Delta\varphi_{AB})\right] = 0$$

For a ***pure*** standing wave, there is also no ***net/time-averaged*** flux of real energy, since energy associated with each of the individual counter-propagating monochromatic plane waves is individually flowing in opposite directions – hence there is no ***net/time-averaged*** flow of energy for a *pure* standing wave.

However, ***instantaneously***, acoustic energy sloshes back and forth during each cycle of oscillation (with period $\tau = 1/f = 2\pi/\omega$) in the longitudinal direction (i.e. the *x*-direction / the propagation direction in this example). In general, the instantaneous flow of acoustic energy has both real and imaginary components, since the instantaneous complex 1-D longitudinal sound intensity has both real and imaginary components, reflected in the fact that the complex *specific* acoustic impedance associated with counter-propagating monochromatic plane waves also has both real and imaginary components. The instantaneous sound energy is "stored" locally in the form of kinetic (u) or potential (p) energy density, however it does not *propagate* anywhere.

The real and imaginary components of the instantaneous 1-D longitudinal sound intensity are:

$$\tilde{I}_{a_{tot}\,r}^{\parallel inst}(x,t) \equiv p_{tot\,r}(x,t)\cdot u_{tot\,r}^{\parallel}(x,t) = \frac{|\tilde{A}|^2}{z_o}\left[\cos^2(\omega t - kx + \varphi_A) - |\tilde{R}|^2\cos^2(\omega t + kx + \varphi_B)\right]$$

$$\tilde{I}_{a_{tot}\,i}^{\parallel inst}(x,t) \equiv p_{tot\,r}(x,t)\cdot u_{tot\,i}^{\parallel}(x,t)$$

$$= \frac{|\tilde{A}|^2}{z_o}\left[\begin{array}{l}\cos(\omega t - kx + \varphi_A)\sin(\omega t - kx + \varphi_A) - |\tilde{R}|^2\sin(\omega t + kx + \varphi_B)\cos(\omega t + kx + \varphi_B)\\ -|\tilde{R}|\cos(\omega t - kx + \varphi_A)\sin(\omega t + kx + \varphi_B) + |\tilde{R}|\sin(\omega t - kx + \varphi_A)\cos(\omega t + kx + \varphi_B)\end{array}\right]$$

The instantaneous total energy density (also a purely real quantity) is:

$$w_{tot}^{inst}(x,t) \equiv w_{potl}^{inst}(x,t) + w_{kin}^{inst}(x,t) = \frac{|\tilde{A}|^2}{z_o c}\left[\cos^2(\omega t - kx + \varphi_A) + |\tilde{R}|^2\cos^2(\omega t + kx + \varphi_B)\right]$$

Thus we see that, here for the case of two counter-propagating monochromatic plane waves that (obviously):

$$\tilde{I}_{a_{tot}}^{\parallel inst}(x,t) \neq c\cdot w_{tot}^{inst}(x,t)\;!!!$$

This is (of course) also true for the ***time-averaged*** versions of these quantities:

$$\left\langle \tilde{I}_{a_{tot}}^{\parallel}(x)\right\rangle_t \equiv \frac{1}{2}\tilde{p}_{tot}(x,t)\cdot\tilde{u}_{tot}^{\parallel *}(x,t) = \frac{1}{2}\frac{|\tilde{A}|^2}{z_o}\left[\left\{1 - |\tilde{R}|^2\right\} + 2i|\tilde{R}|\sin(2kx + \Delta\varphi_{BA})\right]$$

$$\left\langle w_{tot}(x)\right\rangle_t = \frac{1}{2}\frac{|\tilde{A}|^2}{z_o c}\left[1 + |\tilde{R}|^2\right]$$

$$\left\langle \tilde{I}_{a_{tot}}^{\parallel}(x)\right\rangle_t \neq c\left\langle w_{tot}(x)\right\rangle_t$$

The reason for this, is of course due to the fact that the complex 1-D longitudinal sound intensity has a non-zero imaginary component, whereas the energy density is a purely *real* quantity.

These relations can be "rescued" only if c itself is a ***complex***, ***position*** and ***time-dependent*** 3-D ***vector*** quantity, $\vec{\tilde{c}}(\vec{r},t)$. Then we would have the complex relations:

$$\vec{\tilde{I}}_a^{inst}(\vec{r},t) \equiv \vec{\tilde{c}}^{inst}(\vec{r},t)\cdot w_{tot}^{inst}(\vec{r},t) \text{ and } \left\langle \vec{\tilde{I}}_a(\vec{r})\right\rangle_t \equiv \left\langle \vec{\tilde{c}}(\vec{r})\cdot w_{tot}(\vec{r})\right\rangle_t = \left\langle \vec{\tilde{c}}(\vec{r})\right\rangle_t \cdot \left\langle w_{tot}(\vec{r})\right\rangle_t$$

The physical interpretation of real and imaginary components of complex $\vec{\tilde{c}}(\vec{r},t)$ would also be quite restricted, because we ***cannot*** have this $\vec{\tilde{c}}(\vec{r},t)$ connected e.g. to the dispersion relation $c = f\lambda = \omega/k$, because frequency f and/or angular frequency ω are always *real* quantities, and e.g. for dissipationless/lossless/zero-viscosity media, and for e.g. propagation of plane waves in "free" air, this requires *real* wavenumbers k and/or *real* wavelengths λ. Hence $c = f\lambda = \omega/k$ must be purely real for such situations. Hence, $\vec{\tilde{c}}(\vec{r},t)$ physically must mean something else… it is a tracer of the complex propagational *field* nature of the *sound wave itself*, and not of the local spatial/temporal *mechanical* pressure/particle velocity aspects of the *fluid medium*.

For the 1-D situation at hand – i.e. that of counter-propagating monochromatic plane waves, e.g. for the ***time-averaged*** version of this relation, we see that:

$$\left\langle \tilde{I}_a^{||}(x)\right\rangle_t \equiv \left\langle \tilde{c}^{||}(x)\cdot w_{tot}(x)\right\rangle_t = \left\langle \tilde{c}^{||}(x)\right\rangle_t \cdot \left\langle w_{tot}(x)\right\rangle_t$$

is:

$$\frac{1}{2}\frac{\left|\tilde{A}\right|^2}{z_o}\left[\left\{1-\left|\tilde{R}\right|^2\right\}+2i\left|\tilde{R}\right|\sin\left(2kx+\Delta\varphi_{BA}\right)\right] = \frac{1}{2}\frac{\left|\tilde{A}\right|^2}{z_o c}\left[1+\left|\tilde{R}\right|^2\right]\left\langle \tilde{c}^{||}(x)\right\rangle_t$$

therefore:

$$\left\langle \tilde{c}^{||}(x)\right\rangle_t = \left\langle c_r^{||}(x)\right\rangle_t + i\left\langle c_i^{||}(x)\right\rangle_t = c\frac{\left[\left\{1-\left|\tilde{R}\right|^2\right\}+2i\left|\tilde{R}\right|\sin\left(2kx+\Delta\varphi_{BA}\right)\right]}{\left[1+\left|\tilde{R}\right|^2\right]}$$

with:

$$\left\langle c_r^{||}(x)\right\rangle_t = c\left(\frac{1-\left|\tilde{R}\right|^2}{1+\left|\tilde{R}\right|^2}\right) \text{ and: } \left\langle c_i^{||}(x)\right\rangle_t = c\frac{2\left|\tilde{R}\right|\sin\left(2kx+\Delta\varphi_{BA}\right)}{1+\left|\tilde{R}\right|^2}$$

Note that for $\left|\tilde{R}\right| = 1$ (i.e. a "pure" standing wave) for an observer/listener's position at $x = 0$ that:

$$\left\langle c_r^{||}(x=0)\right\rangle_t = 0 \text{ and: } \left\langle c_i^{||}(x=0)\right\rangle_t = c\sin\Delta\varphi_{BA}$$

i.e.

$$\left\langle \tilde{c}^{||}(x=0)\right\rangle_t = \left\langle c_r^{||}(x)\right\rangle_t + i\left\langle c_i^{||}(x)\right\rangle_t = 0 + i\cdot c\sin\Delta\varphi_{BA}$$

We can also define ***real***, instantaneous and time-averaged scalar quantities that respectively represent the ***fractional*** (i.e. ***dimensionless***) amount of the instantaneous and time-averaged total energy density (purely real scalar quantities) associated with the ***real*** (i.e. ***propagating***) and ***imaginary*** (i.e. ***non-propagating***) components of the instantaneous and time-averaged complex 3-D vector acoustic intensities:

$$f_r^{inst}(\vec{r},t) \equiv \frac{c_r^{inst\,2}(\vec{r},t)}{\left|\vec{\tilde{c}}^{inst}(\vec{r},t)\right|^2} \quad \text{and:} \quad f_i^{inst}(\vec{r},t) \equiv \frac{c_i^{inst\,2}(\vec{r},t)}{\left|\vec{\tilde{c}}^{inst}(\vec{r},t)\right|^2}$$

with:

$$f_r^{inst}(\vec{r},t) + f_i^{inst}(\vec{r},t) = \frac{c_r^{inst\,2}(\vec{r},t) + c_i^{inst\,2}(\vec{r},t)}{\left|\vec{\tilde{c}}^{inst}(\vec{r},t)\right|^2} = 1$$

and:

$$\left\langle f_r(\vec{r})\right\rangle_t \equiv \frac{\left\langle c_r^2(\vec{r})\right\rangle_t}{\left\langle\left|\vec{\tilde{c}}(\vec{r})\right|^2\right\rangle_t} \quad \text{and:} \quad \left\langle f_i(\vec{r})\right\rangle_t \equiv \frac{\left\langle c_i^2(\vec{r})\right\rangle_t}{\left\langle\left|\vec{\tilde{c}}(\vec{r})\right|^2\right\rangle_t}$$

with:

$$\left\langle f_r(\vec{r})\right\rangle_t + \left\langle f_i(\vec{r})\right\rangle_t = \frac{\left\langle c_r^2(\vec{r})\right\rangle_t + \left\langle c_i^2(\vec{r})\right\rangle_t}{\left\langle\left|\vec{\tilde{c}}(\vec{r})\right|^2\right\rangle_t} = 1$$

We hasten to add, that while these quantities may at first sight, seem rather startling, they really are not… since: $\vec{\tilde{I}}_a^{inst}(\vec{r},t) \equiv \vec{\tilde{c}}^{inst}(\vec{r},t)\cdot w_{tot}^{inst}(\vec{r},t)$ and: $\left\langle \vec{\tilde{I}}_a^{\parallel}(\vec{r})\right\rangle_t \equiv \left\langle \tilde{c}^{\parallel}(\vec{r})\right\rangle_t \cdot \left\langle w_{tot}(\vec{r})\right\rangle_t$, then:

$$\vec{\tilde{c}}^{inst}(\vec{r},t) \equiv \vec{\tilde{I}}_a^{inst}(\vec{r},t) \Big/ w_{tot}^{inst}(\vec{r},t) \quad \text{and:} \quad \left\langle \tilde{c}^{\parallel}(\vec{r})\right\rangle_t \equiv \left\langle \vec{\tilde{I}}_a^{\parallel}(\vec{r})\right\rangle_t \Big/ \left\langle w_{tot}(\vec{r})\right\rangle_t$$

and therefore:

$$f_r^{inst}(\vec{r},t) \equiv \frac{c_r^{inst\,2}(\vec{r},t)}{\left|\vec{\tilde{c}}^{inst}(\vec{r},t)\right|^2} = \frac{I_{a\,r}^{inst\,2}(\vec{r},t)}{\left|\vec{\tilde{I}}_a^{inst}(\vec{r},t)\right|^2} \quad \text{and:} \quad f_i^{inst}(\vec{r},t) \equiv \frac{c_i^{inst\,2}(\vec{r},t)}{\left|\vec{\tilde{c}}^{inst}(\vec{r},t)\right|^2} = \frac{I_{a\,i}^{inst\,2}(\vec{r},t)}{\left|\vec{\tilde{I}}_a^{inst}(\vec{r},t)\right|^2}$$

with:

$$f_r^{inst}(\vec{r},t) + f_i^{inst}(\vec{r},t) = \frac{c_r^{inst\,2}(\vec{r},t) + c_i^{inst\,2}(\vec{r},t)}{\left|\vec{\tilde{c}}^{inst}(\vec{r},t)\right|^2} = \frac{I_{a\,r}^{inst\,2}(\vec{r},t) + I_{a\,i}^{inst\,2}(\vec{r},t)}{\left|\vec{\tilde{I}}_a^{inst}(\vec{r},t)\right|^2} = 1$$

and:

$$\left\langle f_r(\vec{r})\right\rangle_t \equiv \frac{\left\langle c_r^2(\vec{r})\right\rangle_t}{\left\langle\left|\vec{\tilde{c}}(\vec{r})\right|^2\right\rangle_t} = \frac{\left\langle I_{a\,r}^2(\vec{r})\right\rangle_t}{\left\langle\left|\vec{\tilde{I}}_a(\vec{r})\right|^2\right\rangle_t} \quad \text{and:} \quad \left\langle f_i(\vec{r})\right\rangle_t \equiv \frac{\left\langle c_i^2(\vec{r})\right\rangle_t}{\left\langle\left|\vec{\tilde{c}}(\vec{r})\right|^2\right\rangle_t} = \frac{\left\langle I_{a\,i}^2(\vec{r})\right\rangle_t}{\left\langle\left|\vec{\tilde{I}}_a(\vec{r})\right|^2\right\rangle_t}$$

with:

$$\left\langle f_r(\vec{r})\right\rangle_t + \left\langle f_i(\vec{r})\right\rangle_t = \frac{\left\langle c_r^2(\vec{r})\right\rangle_t + \left\langle c_i^2(\vec{r})\right\rangle_t}{\left\langle\left|\vec{\tilde{c}}(\vec{r})\right|^2\right\rangle_t} = \frac{\left\langle I_{a\,r}^2(\vec{r})\right\rangle_t + \left\langle I_{a\,i}^2(\vec{r})\right\rangle_t}{\left\langle\left|\vec{\tilde{I}}_a(\vec{r})\right|^2\right\rangle_t} = 1$$

For the 1-D situation at hand – i.e. that of counter-propagating monochromatic plane waves, e.g. for the ***time-averaged*** version of these relations, we see that:

$$\left\langle f_r^{\parallel}(x)\right\rangle_t \equiv \frac{\left\langle c_r^{\parallel 2}(x)\right\rangle_t}{\left\langle\left|\tilde{c}^{\parallel}(x)\right|^2\right\rangle_t} = \frac{\left\langle I_{ar}^2(x)\right\rangle_t}{\left\langle\left|\vec{\tilde{I}}_a(x)\right|^2\right\rangle_t} = \frac{\left\{1-\left|\tilde{R}\right|^2\right\}^2}{\left\{1-\left|\tilde{R}\right|^2\right\}^2 + 4\left|\tilde{R}\right|^2\sin^2\left(2kx+\Delta\varphi_{BA}\right)}$$

and:

$$\left\langle f_i^{\parallel}(x)\right\rangle_t \equiv \frac{\left\langle c_i^{\parallel 2}(x)\right\rangle_t}{\left\langle\left|\vec{\tilde{c}}^{\parallel}(x)\right|^2\right\rangle_t} = \frac{\left\langle I_{ai}^2(x)\right\rangle_t}{\left\langle\left|\vec{\tilde{I}}_a(x)\right|^2\right\rangle_t} = \frac{4\left|\tilde{R}\right|^2\sin^2\left(2kx+\Delta\varphi_{BA}\right)}{\left\{1-\left|\tilde{R}\right|^2\right\}^2 + 4\left|\tilde{R}\right|^2\sin^2\left(2kx+\Delta\varphi_{BA}\right)}$$

with:

$$\left\langle f_r^{\parallel}(x)\right\rangle_t + \left\langle f_i^{\parallel}(x)\right\rangle_t = \frac{\left\langle c_r^{\parallel 2}(x)\right\rangle_t + \left\langle c_i^{\parallel 2}(x)\right\rangle_t}{\left\langle\left|\tilde{c}^{\parallel}(x)\right|^2\right\rangle_t} = \frac{\left\langle I_{ar}^2(x)\right\rangle_t + \left\langle I_{ai}^2(x)\right\rangle_t}{\left\langle\left|\vec{\tilde{I}}_a(x)\right|^2\right\rangle_t} = 1$$

Note that for $\left|\tilde{R}\right| = 1$ (i.e. a "pure" standing wave) and an arbitrary observer's position x that:

$$\left\langle f_r^{\parallel}(x)\right\rangle_t \equiv \frac{\left\langle c_r^{\parallel 2}(x)\right\rangle_t}{\left\langle\left|\tilde{c}^{\parallel}(x)\right|^2\right\rangle_t} = \frac{\left\langle I_{ar}^2(x)\right\rangle_t}{\left\langle\left|\vec{\tilde{I}}_a(x)\right|^2\right\rangle_t} = 0 \quad \text{and:} \quad \left\langle f_i^{\parallel}(x)\right\rangle_t \equiv \frac{\left\langle c_i^{\parallel 2}(x)\right\rangle_t}{\left\langle\left|\vec{\tilde{c}}^{\parallel}(x)\right|^2\right\rangle_t} = \frac{\left\langle I_{ai}^2(x)\right\rangle_t}{\left\langle\left|\vec{\tilde{I}}_a(x)\right|^2\right\rangle_t} = 1$$

with:

$$\left\langle f_r^{\parallel}(x)\right\rangle_t + \left\langle f_i^{\parallel}(x)\right\rangle_t = \frac{\left\langle c_r^{\parallel 2}(x)\right\rangle_t + \left\langle c_i^{\parallel 2}(x)\right\rangle_t}{\left\langle\left|\tilde{c}^{\parallel}(x)\right|^2\right\rangle_t} = \frac{\left\langle I_{ar}^2(x)\right\rangle_t + \left\langle I_{ai}^2(x)\right\rangle_t}{\left\langle\left|\vec{\tilde{I}}_a(x)\right|^2\right\rangle_t} = 1$$

The physical meaning of the complex quantity $\tilde{R} \equiv \tilde{B}/\tilde{A} = \left|\tilde{R}\right| e^{i\Delta\varphi_{BA}}$ used in (all of) the above formulae for this counter-propagating plane wave problem can also be used to describe various other types of acoustical physics situations, e.g. interpreting $\tilde{R}$ as the complex acoustic ***reflectance*** associated with a sound wave reflecting off of a surface.

The ***reflection*** ***coefficient*** associated with the surface is then defined as: $0 \le \mathbb{R} \equiv \left|\tilde{R}\right|^2 \le 1$.

If a sound wave is only partially reflected from a surface, then it is either partially transmitted (with complex acoustic ***transmittance*** $\tilde{T}$ and corresponding ***transmission*** ***coefficient*** $0 \le \mathbb{T} \equiv \left|\tilde{T}\right|^2 \le 1$) and/or is absorbed by the surface (with complex acoustic ***absorbance*** $\tilde{A}$ and corresponding ***absorption*** ***coefficient*** $0 \le \mathbb{A} \equiv \left|\tilde{A}\right|^2 \le 1$), since we must have: $\mathbb{R} + \mathbb{T} + \mathbb{A} = 1$.

Limiting/Special Cases of Interest:

1.) A single monochromatic traveling plane wave (emitted from a sound source e.g. located at $x = -\infty$) propagating in the +*ve* *x*-direction and reflects, at normal incidence, off of a rigid, perfectly reflecting infinite plane (located at $x = x_o > 0$), thereby producing a reflected wave (of equal amplitude) that propagates in the –*ve* *x*-direction. This situation physically corresponds to $\tilde{R} = |\tilde{R}|e^0 = +1$ at $x = x_o > 0$, which has the associated boundary condition $\tilde{p}_{refl}(x = x_o, t) = \tilde{p}_{inc}(x = x_o, t)$, i.e. no phase change upon reflection, such that an over-pressure ***anti-node*** exists at $x = x_o > 0$:

$$\tilde{p}_{tot}(x = x_o, t) = \tilde{p}_{inc}(x = x_o, t) + \tilde{p}_{refl}(x = x_o, t) = 2\tilde{p}_{inc}(x = x_o, t).$$

2.) A single monochromatic traveling plane wave (emitted from a sound source e.g. located at $x = -\infty$) propagating in the +*ve* *x*-direction and reflects, at normal incidence, off of an infinite ***pressure-release*** plane consisting of an air-water interface (located at $x = x_o > 0$), thereby producing a reflected wave (of equal amplitude) that propagates in the –*ve* *x*-direction.

This situation physically corresponds to $\tilde{R} = |\tilde{R}|e^{i\pi} = -1$. An air-water interface (viewed from the water side) closely approximates an *ideal* ***pressure-release surface***, for which the boundary condition at the pressure-release surface is $\tilde{p}_{refl}(x = x_o, t) = -\tilde{p}_{inc}(x = x_o, t)$ (i.e. a phase change of 180° upon reflection), such that an over-pressure *node* exists at $x = x_o > 0$:

$$\tilde{p}_{tot}(x = x_o, t) = \tilde{p}_{inc}(x = x_o, t) - \tilde{p}_{refl}(x = x_o, t) = 0.$$

3.) The most general case: A single monochromatic traveling plane wave (emitted from a sound source e.g. located at $x = -\infty$) propagating in the +*ve* *x*-direction and reflects, at normal incidence off of an infinite plane (located at $x = x_o > 0$) of *arbitrary* characteristics – e.g. it could be a "*passive*" surface that is only partially *reflecting/partially absorbing* (hence $|\tilde{R}| < 1$) and in principle could have associated with it e.g. a frequency-dependent phase shift upon reflection $-\pi \le \Delta\varphi_{BA}(x = x_o, \omega) \le \pi$, thereby producing a reflected wave that propagates in the –*ve* *x*-direction. This situation physically corresponds to the most general $\tilde{R} = |\tilde{R}|e^{i\Delta\varphi_{BA}}$. If the reflecting surface were "*active*", it is also possible that $|\tilde{R}| > 1$ (!), and depending on the details of the response of the "active" reflecting surface, the phase shift could be $-\pi \le \Delta\varphi_{BA}(x = x_o, \omega) \le \pi$.

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