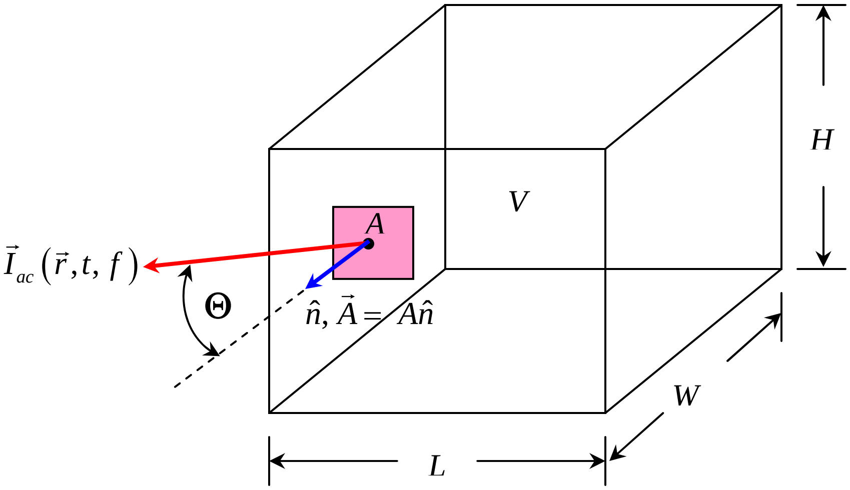
Derivation of the Sabine Equation: Conservation of Energy

Consider a room of volume $V = H \times W \times L$ (m^3) with perfectly reflecting walls, filled with a uniform, steady-state (*i.e.* equilibrium) acoustic energy density at given frequency f (*Hz*) within the volume V of the room. Uniform means that a given time t: $w_{ac}(\vec{r}, t, f) = w_{ac}(t, f) = constant$ (*SI* units: *Joules*/m^3). The room also has a small opening of area A (m^2) in it, as shown in the figure below:



In the steady-state, the rate of acoustical energy input e.g. by a point sound source within the room equals the rate at which acoustical energy is "leaking" out of the room through the hole of area A, i.e. the acoustical power input by the sound source in the room into the room = the acoustical power leaving the room through the hole of area A. In this idealized model of a room with perfectly reflecting walls, the hole of area A thus represents absorption of sound in a real room with finite reflectivity walls, i.e. walls that have some absorption associated with them.

Suppose at time $t = 0$ the sound source in the room {located far from the hole} is turned off. Since the sound energy density is uniform in the room, the sound energy contained in the room $W_{ac}(t, f) = \int_V w_{ac}(\vec{r}, t, f)\, d^3r = w_{ac}(t, f) \int_V d^3r = w_{ac}(t, f) V$ will thus decrease with time, since acoustical energy is (slowly) leaking out of the room through the opening of area A.

The instantaneous acoustical power at the frequency f passing through the hole of area A is the instantaneous time-rate of change of the acoustic energy in the room, i.e.

$$P_{ac}(t, f) = \frac{\partial W_{ac}(t, f)}{\partial t}$$

However, the instantaneous acoustical power loss at the frequency f associated with the flux of acoustic energy passing through the hole of area A is also $P_{ac}(t, f) = -\int_A \vec{I}_{ac}(\vec{r}, t, f) \bullet d\vec{A}$ where $\vec{I}_{ac}(\vec{r}, t, f)$ is the instantaneous 3-D vector sound intensity at the point $\vec{r}$ at frequency f (*SI* units: *Watts*/m^2) and $d\vec{A} = dA\hat{n}$ is a infinitesimal vector area element associated with the hole of area A, and $\hat{n}$ is the outward-pointing unit normal to the hole of area A, as shown in the above figure.

Thus:

$$P_{ac}(t,f)=\frac{\partial W_{ac}(t,f)}{\partial t}=-\int_A \vec{I}_{ac}(\vec{r},t,f)\bullet d\vec{A}=-\int_A I_{ac}(t,f)\cos\Theta(t)\,dA$$

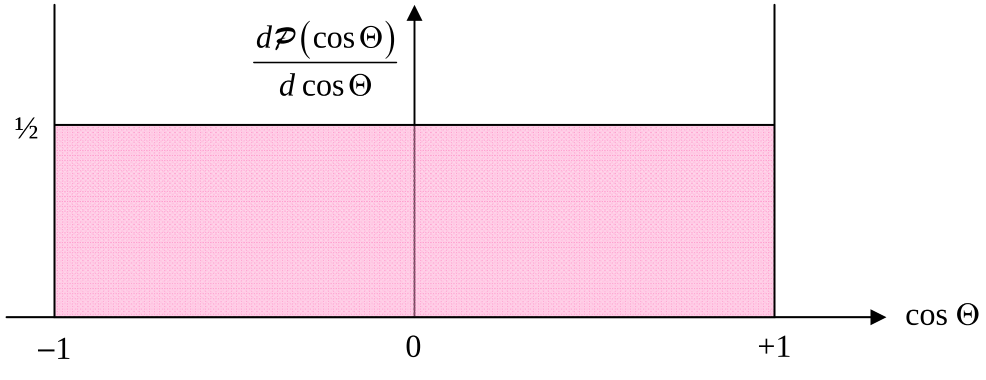
where $\cos\Theta(t)$ is the instantaneous direction cosine / $\Theta(t)$ is the instantaneous 3-D opening angle between the two vectors $\vec{I}_{ac}(t,f)$ and $\vec{A}$ (as shown in the figure above), since $\vec{I}_{ac}(\vec{r},t,f)\bullet\hat{n}=I_{ac}(\vec{r},t,f)\cos\Theta$.

In the steady-state, the *magnitude* of the 3-D vector sound intensity is constant in time at any given point $\vec{r}$ inside the volume *V* of the room, or on/at the opening of the hole of area *A*, thus $\left|\vec{I}_{ac}(\vec{r},t,f)\right|=I_{ac}(\vec{r},f)$, however the *direction* of the 3-D vector sound intensity at any given point $\vec{r}$ fluctuates randomly from one moment to the next. At/on the surface of the opening of the hole of area *A*, the direction of the 3-D vector sound intensity associated with energy leaking out of the room of volume *V* is such that the direction of the 3-D sound intensity points randomly from moment-to-moment in the forward-going hemisphere, i.e. is contained within a solid angle $d\Omega$ associated only with the forward half of 4π steradians (since sound energy is leaking out of the room, i.e. energy is not coming into the room from the outside).

We are not interested in following the instantaneous, moment-to-moment/short-time scale fluctuations in the 3-D vector sound intensity $\vec{I}_{ac}(\vec{r},t,f)$, but we are interested in the mean power loss, averaged over these moment-to-moment fluctuations. For randomly fluctuating direction in $\vec{I}_{ac}(\vec{r},t,f)$, the mean power loss through the hole of area *A*, averaging over such moment-to-moment fluctuations is:

$$\left\langle P_{ac}(t,f)\right\rangle=\frac{\partial\left\langle W_{ac}(t,f)\right\rangle}{\partial t}=-\left\langle\int_A \vec{I}_{ac}(\vec{r}_{hole},t,f)\bullet d\vec{A}\right\rangle=-\int_A I_{ac}(\vec{r}_{hole},t,f)\left\langle\cos\Theta(t)\right\rangle dA$$

The random, fluctuating moment-to-moment direction in the 3-D vector sound intensity $\vec{I}_{ac}(\vec{r}_{hole},t,f)$ means that $\cos\Theta(t)$ is also random at the hole opening of area *A*. What this means physically is that the probability density distribution $d\mathcal{P}(\cos\Theta)/d\cos\Theta=1/2$ is flat/uniform in the $\cos\Theta$ variable, as shown in the figure below:



Since probability is conserved, we must have:

$$\int_{-1}^{+1}\left(\frac{d\mathcal{P}(\cos\Theta)}{d\cos\Theta}\right)d\cos\Theta=\int_{-1}^{+1}\left(\frac{d\mathcal{P}(x)}{dx}\right)dx=\int_{-1}^{+1}\frac{1}{2}dx=\frac{1}{2}\int_{-1}^{+1}dx=\frac{1}{2}x\Big|_{-1}^{+1}=\frac{1}{2}\cdot 2=1$$

However, for our physical situation <u>here</u>, only the <u>forward</u> half of this probability distribution is occupied $(0\le\cos\Theta\le 1)$ - {sound energy is leaking out of the hole, not into it}, thus:

$$\langle\cos\Theta(t)\rangle=\int_0^1\left(\frac{d\mathcal{P}(\cos\Theta)}{d\cos\Theta}\right)\cos\Theta\,d\cos\Theta=\int_0^1\left(\frac{1}{2}\right)\cos\Theta\,d\cos\Theta=\frac{1}{2}\int_0^1 x\,dx=\frac{1}{2}\cdot\frac{1}{2}x^2\Big|_0^1=\frac{1}{4}$$

and hence:

$$\begin{aligned}\langle P_{ac}(t,f)\rangle=\frac{\partial\langle W_{ac}(t,f)\rangle}{\partial t}&=-\left\langle\int_A\vec{I}_{ac}(\vec{r},t,f)\bullet d\vec{A}\right\rangle=-\int_A I_{ac}(\vec{r},f)\langle\cos\Theta(t)\rangle dA\\&=-\frac{1}{4}\int_A I_{ac}(\vec{r},f)dA=-\frac{1}{4}I_{ac}(\vec{r}_{hole},f)\int_A dA=-\frac{1}{4}I_{ac}(\vec{r}_{hole},f)A\end{aligned}$$

<u>Clarificational Note</u>: In the <u>steady-state</u>, the time interval Δt_{avg} needed for averaging over the moment-to-moment fluctuations in the instantaneous <u>direction</u> of the 3-D sound intensity $\vec{I}_{ac}(\vec{r},t,f)$ is still <u>much</u> less than the characteristic time constant τ associated with sound energy leaking out of the room of volume *V* through the hole of area *A*, i.e. $\Delta t_{avg}\ll\tau$.

Now, recall that the instantaneous 3-D vector sound intensity $\vec{I}_{ac}(\vec{r},t,f)$ is related to the instantaneous scalar acoustic energy density $w_{ac}(\vec{r},t,f)$ (*Joules*/m^3) by the relation $\vec{I}_{ac}(\vec{r},t,f)=\vec{c}w_{ac}(\vec{r},t,f)$ where $\vec{c}$ = velocity vector associated with propagation of sound in air with $|\vec{c}|$ = 343 *m*/*s* at *NTP*. Thus, from the above discussion on averaging out random, moment-to-moment fluctuations in the <u>direction</u> of 3-D sound intensity at the hole of area *A*, we see that:

$$\langle\vec{I}_{ac}(\vec{r},t,f)\rangle=\langle\vec{c}w_{ac}(\vec{r},t,f)\rangle\Rightarrow\langle I_{ac}(\vec{r}_{hole},t,f)\rangle=c\langle w_{ac}(\vec{r}_{hole},t,f)\rangle$$

Note also that at in the <u>steady-state</u>, at time *t* the acoustic energy $W_{ac}(t,f)$ contained within the room of volume *V* is related to the {uniform} acoustic energy density $w_{ac}(\vec{r},t,f)$ by:

$$W_{ac}(t,f)=\int_V w_{ac}(\vec{r},t,f)d^3r=w_{ac}(t,f)\int_V d^3r=w_{ac}(t,f)V$$

Since the energy leaking out of the hole comes from inside the room, by energy conservation, we see that:

$$\begin{aligned}\langle P_{ac}(t,f)\rangle=\frac{\partial\langle W_{ac}(t,f)\rangle}{\partial t}&=-\left\langle\int_A\vec{I}_{ac}(\vec{r},t,f)\bullet d\vec{A}\right\rangle=-\frac{1}{4}\langle I_{ac}(\vec{r}_{hole},t,f)\rangle A=-\frac{1}{4}c\langle w_{ac}(\vec{r}_{hole},t,f)\rangle A\\&=-\frac{cA}{4V}\langle W_{ac}(t,f)\rangle\end{aligned}$$

or: $$\frac{\partial\left\langle W_{ac}(t,f)\right\rangle}{\partial t} = -\frac{cA}{4V}\cdot\left\langle W_{ac}(t,f)\right\rangle = -\frac{1}{\tau}\left\langle W_{ac}(t,f)\right\rangle$$

where we have defined the characteristic time constant $\tau \equiv \frac{4V}{cA}$ (*SI* units: *seconds*).

The equation $\frac{\partial\left\langle W_{ac}(t,f)\right\rangle}{\partial t} = -\frac{1}{\tau}\left\langle W_{ac}(t,f)\right\rangle$ is a linear, first-order homogeneous differential equation {the diffusion, or heat equation} which, for our situation/our initial conditions (at $t = 0$) has the well-known solution of the form:

$$\left\langle W_{ac}(t,f)\right\rangle = \left\langle W_{ac}^{o}(f)\right\rangle e^{-t/\tau}$$

where $\left\langle W_{ac}^{o}(f)\right\rangle$ is the acoustic energy contained in the room at the frequency f at time $t = 0$.

Thus, at time $t = \tau$: $\left\langle W_{ac}(f,t=\tau)\right\rangle = \left\langle W_{ac}^{o}(f)\right\rangle e^{-\tau/\tau} = \left\langle W_{ac}^{o}(f)\right\rangle e^{-1}$ i.e. the acoustic energy at frequency f decreases to $1/e = 1/2.7183 = 0.3679$ of its initial value after a time interval $t = \tau$.

Since $\left\langle I_{ac}(f,t)\right\rangle = c\left\langle w_{ac}(f,t)\right\rangle = c\left\langle W_{ac}(f,t)\right\rangle/V$ we can equivalently rewrite the solution for the acoustic energy in terms of sound intensity as $\left\langle I_{ac}(t,f)\right\rangle = \left\langle I_{ac}^{o}(f)\right\rangle e^{-t/\tau}$ where $\left\langle I_{ac}^{o}(f)\right\rangle$ is the {magnitude} of the sound intensity at the frequency f at time $t = 0$, and instead ask: how long does it take for the sound intensity to decay to one-millionth (10^{-6}) of its initial value, i.e. what is the reverberation time, T_{60}? This occurs when:

$$\left\langle I_{ac}(t=T_{60},f)\right\rangle = \left\langle I_{ac}^{o}(f)\right\rangle e^{-T_{60}/\tau} = 10^{-6}\left\langle I_{ac}^{o}(f)\right\rangle$$

i.e. when $e^{-T_{60}/\tau} = 10^{-6}$. Take the natural log of both sides of this relation: $\ln\left(e^{-T_{60}/\tau}\right) = \ln\left(10^{-6}\right)$.

But $\ln\left(e^{-T_{60}/\tau}\right) = -T_{60}/\tau$. Thus: $-T_{60}/\tau = \ln\left(10^{-6}\right)$ or $T_{60} = -\tau\ln\left(10^{-6}\right)$ and since $\tau \equiv \frac{4V}{cA}$, we thus find that the reverberation time T_{60} is:

$$T_{60} = \left\{-\frac{4\cdot\ln\left(10^{-6}\right)}{c}\right\}\frac{V}{A} = \kappa\frac{V}{A}$$

The numerical value of the "universal" constant, κ is:

$$\kappa = -\frac{4\cdot\ln\left(10^{-6}\right)}{c} = +\frac{4\cdot 13.8155}{343\,m/s} = +\frac{55.262}{343\,m/s} = 0.1611\,s/m\left(=0.049\,s/ft\right)$$

Thus, the Sabine equation is: $\boxed{T_{60} = 0.161\frac{V}{A}(\textit{metric units}) = 0.049\frac{V}{A}(\textit{english units})}$.

We also see that $\boxed{T_{60} = -\tau\ln\left(10^{-6.0}\right) = 13.8155\,\tau}$ and $\boxed{T_{30} = -\tau\ln\left(10^{-3.0}\right) = 6.9078\,\tau = \tfrac{1}{2}T_{60}}$.