

## LECTURE NOTES 7

### ELECTROMAGNETIC WAVES IN CONDUCTORS

Inside a conductor, free charges can move/migrate around in response to  $EM$  fields contained therein, as we saw for the case of the longitudinal  $\vec{E}$ -field inside a current-carrying wire that had a potential difference  $\Delta V$  across its ends. Even in the static case of electric charge residing on the surface of a conductor, we saw that  $\vec{E}_{inside}(\vec{r}) = 0$ , but recall that this actually means (as we showed last semester) that the NET electric field inside the conductor is zero, i.e.  $\vec{E}_{inside}^{NET}(\vec{r}) = 0$ .

n.b. here, we assume {for simplicity's sake} that the conductor is linear/homogeneous/isotropic – i.e. no crystalline structure/no anisotropies/no inhomogenities/no non-uniformities/no voids/no defects...

From Ohm's Law, we know that the free current density is proportional to the (ambient) electric field inside the conductor:  $\vec{J}_{free}(\vec{r}, t) = \sigma_c \vec{E}(\vec{r}, t)$  where  $\sigma_c =$  conductivity of the metal conductor (Siemens/m =  $\text{Ohm}^{-1}/\text{m}$ ) and  $\sigma_c = 1/\rho_c$  where  $\rho_c =$  resistivity of the metal conductor (Ohm-m).

Thus, inside a conductor  $\vec{J}_{free}(\vec{r}, t) \neq 0$ , we assume that the linear/homogeneous/isotropic conducting medium has electric permittivity  $\epsilon$  and magnetic permeability  $\mu$ . Maxwell's equations inside such a conductor are thus:

$$\begin{array}{ll}
 1) \quad \vec{\nabla} \cdot \vec{E}(\vec{r}, t) = \rho_{free}(\vec{r}, t) / \epsilon & 2) \quad \vec{\nabla} \cdot \vec{B}(\vec{r}, t) = 0 \\
 3) \quad \vec{\nabla} \times \vec{E}(\vec{r}, t) = -\frac{\partial \vec{B}(\vec{r}, t)}{\partial t} & 4) \quad \vec{\nabla} \times \vec{B}(\vec{r}, t) = \mu \vec{J}_{free}(\vec{r}, t) + \mu \epsilon \frac{\partial \vec{E}(\vec{r}, t)}{\partial t} = \mu \sigma_c \vec{E}(\vec{r}, t) + \mu \epsilon \frac{\partial \vec{E}(\vec{r}, t)}{\partial t}
 \end{array}$$

Using Ohm's Law:  
 $\vec{J}_{free}(\vec{r}, t) = \sigma_c \vec{E}(\vec{r}, t)$

Electric charge is (always) conserved, thus the continuity equation inside the conductor is:

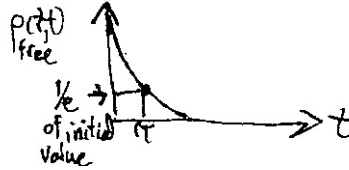
$$\begin{array}{l}
 \vec{\nabla} \cdot \vec{J}_{free}(\vec{r}, t) = -\frac{\partial \rho_{free}(\vec{r}, t)}{\partial t} \quad \text{but:} \quad \vec{J}_{free}(\vec{r}, t) = \sigma_c \vec{E}(\vec{r}, t) \\
 \therefore \sigma_c (\vec{\nabla} \cdot \vec{E}(\vec{r}, t)) = -\frac{\partial \rho_{free}(\vec{r}, t)}{\partial t} \quad \text{but:} \quad \vec{\nabla} \cdot \vec{E}(\vec{r}, t) = \rho_{free}(\vec{r}, t) / \epsilon \\
 \text{thus:} \quad \frac{\sigma_c \rho_{free}(\vec{r}, t)}{\epsilon} = -\frac{\partial \rho_{free}(\vec{r}, t)}{\partial t} \quad \text{or:} \quad \frac{\partial \rho_{free}(\vec{r}, t)}{\partial t} + \left( \frac{\sigma_c}{\epsilon} \right) \rho_{free}(\vec{r}, t) = 0 \quad \leftarrow \begin{array}{l} 1^{\text{st}} \text{ order linear,} \\ \text{homogeneous} \\ \text{differential equation} \end{array}
 \end{array}$$

The {physical} general solution of this differential equation for the free charge density is of the form:

$$\rho_{free}(\vec{r}, t) = \rho_{free}(\vec{r}, t=0) e^{-\sigma_c t / \epsilon} \quad \text{i.e. a damped exponential!!!}$$

Thus, the continuity equation  $\vec{\nabla} \cdot \vec{J}_{free}(\vec{r}, t) = -\partial \rho_{free}(\vec{r}, t) / \partial t$  inside a conductor tells us that any free charge density  $\rho_{free}(\vec{r}, t=0)$  initially present at time  $t=0$  is exponentially damped / dissipated in a characteristic time  $\tau_{relax} \equiv \epsilon / \sigma_c =$  charge relaxation time {aka time constant}, such that:

$$\rho_{free}(\vec{r}, t) = \rho_{free}(\vec{r}, t=0) e^{-\sigma_c t / \epsilon} = \rho_{free}(\vec{r}, t=0) e^{-t / \tau_{relax}}$$



Calculation of the Charge Relaxation Time for Pure Copper:

$$\rho_{Cu} = 1 / \sigma_{Cu} = 1.68 \times 10^{-8} \Omega \cdot m \Rightarrow \sigma_{Cu} = 1 / \rho_{Cu} = 5.95 \times 10^7 \text{ Siemens/m}$$

If we assume  $\epsilon_{Cu} \approx 3\epsilon_o = 3 \times 8.85 \times 10^{-8} \text{ F/m}$  for copper metal, then:

$$\tau_{Cu}^{relax} = \epsilon_{Cu} / \sigma_{Cu} = \rho_{Cu} \epsilon_{Cu} = 4.5 \times 10^{-19} \text{ sec} \quad !!!$$

However, recall that the characteristic/mean collision time of free electrons in pure copper is  $\tau_{Cu}^{coll} \approx \lambda_{Cu}^{coll} / v_{thermal}^{Cu}$  where  $\lambda_{Cu}^{coll} \approx 3.9 \times 10^{-8} \text{ m}$  = mean free path (between successive collisions) in pure copper, and  $v_{thermal}^{Cu} \approx \sqrt{3k_B T / m_e} \approx 12 \times 10^5 \text{ m/sec}$  and thus we obtain  $\tau_{coll}^{Cu} \approx 3.2 \times 10^{-13} \text{ sec}$ .

Hence we see that the calculated charge relaxation time in pure copper,  $\tau_{Cu}^{relax} \approx 4.5 \times 10^{-19} \text{ sec}$  is  $\ll$  than the calculated collision time in pure copper,  $\tau_{coll}^{Cu} \approx 3.2 \times 10^{-13} \text{ sec}$ .

Furthermore, the experimentally measured charge relaxation time in pure copper is  $\tau_{Cu}^{relax}(\text{exp't}) \approx 4.0 \times 10^{-14} \text{ sec}$ , which is  $\approx 5$  orders of magnitude larger than the calculated charge relaxation time  $\tau_{Cu}^{relax} \approx 4.5 \times 10^{-19} \text{ sec}$ . The problem here is that {the macroscopic} Ohm's Law is simply out of its range of validity on such short time scales! Two additional facts here are that both  $\epsilon$  and  $\sigma_c$  are frequency-dependent quantities {i.e.  $\epsilon = \epsilon(\omega)$  and  $\sigma_c = \sigma_c(\omega)$ }, which becomes increasingly important at the higher frequencies ( $f = 2\pi/\omega \sim 1/\tau_{relax}$ ) associated with short time-scale, transient-type phenomena!

So in reality, if we are willing to wait even a short time (e.g.  $\Delta t \sim 1 \text{ ps} = 10^{-12} \text{ sec}$ ) then any initial free charge density  $\rho_{free}(\vec{r}, t=0)$  accumulated inside the conductor at  $t=0$  will have dissipated away/damped out, and from that time onwards,  $\rho_{free}(\vec{r}, t) = 0$  can be safely assumed.

Thus, after many charge relaxation time constants, e.g.  $20\tau^{relax} \leq \Delta t \approx 1 \text{ ps} = 10^{-12} \text{ sec}$ , then Maxwell's equations for a conductor become {with  $\rho_{free}(\vec{r}, t \geq \Delta t) = 0$  from then onwards}:

$$\begin{array}{ll}
 1) \quad \boxed{\vec{\nabla} \cdot \vec{E}(\vec{r}, t) = 0} & 2) \quad \boxed{\vec{\nabla} \cdot \vec{B}(\vec{r}, t) = 0} & \boxed{\text{Maxwell's equations for a } \underline{\text{charge-equilibrated}} \text{ conductor}} \\
 3) \quad \boxed{\vec{\nabla} \times \vec{E}(\vec{r}, t) = -\frac{\partial \vec{B}(\vec{r}, t)}{\partial t}} & 4) \quad \boxed{\vec{\nabla} \times \vec{B}(\vec{r}, t) = \mu\sigma_c \vec{E}(\vec{r}, t) + \mu\epsilon \frac{\partial \vec{E}(\vec{r}, t)}{\partial t} = \mu \left( \sigma_c \vec{E}(\vec{r}, t) + \epsilon \frac{\partial \vec{E}(\vec{r}, t)}{\partial t} \right)}
 \end{array}$$

Now because these equations are different {n.b. only equation 4) has changed} from the previous derivation(s) of monochromatic plane *EM* waves propagating in free space/vacuum and/or in linear/homogeneous/isotropic non-conducting materials, we re-derive the wave equations for  $\vec{E}$  &  $\vec{B}$  from scratch. As before, we apply  $\vec{\nabla} \times ( )$  to equations 3) and 4):

$$\begin{array}{ll}
 \boxed{\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\frac{\partial}{\partial t} (\vec{\nabla} \times \vec{B})} & \boxed{\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \mu \left( \sigma_c (\vec{\nabla} \times \vec{E}) \right) + \epsilon \frac{\partial}{\partial t} (\vec{\nabla} \times \vec{E})} \\
 = \boxed{\vec{\nabla} \left( \cancel{\vec{Y} \cdot \vec{E}} \right) - \nabla^2 \vec{E} = -\frac{\partial}{\partial t} \left( \mu\sigma_c \vec{E} + \mu\epsilon \frac{\partial \vec{E}}{\partial t} \right)} & = \boxed{\vec{\nabla} \left( \cancel{\vec{Y} \cdot \vec{B}} \right) - \nabla^2 \vec{B} = -\mu\sigma_c \frac{\partial \vec{B}}{\partial t} - \mu\epsilon \frac{\partial^2 \vec{B}}{\partial t^2}} \\
 = \boxed{\nabla^2 \vec{E} = \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2} + \mu\sigma_c \frac{\partial \vec{E}}{\partial t}} & = \boxed{\nabla^2 \vec{B} = \mu\epsilon \frac{\partial^2 \vec{B}}{\partial t^2} + \mu\sigma_c \frac{\partial \vec{B}}{\partial t}} \\
 \underline{\text{Again:}} \quad \boxed{\nabla^2 \vec{E}(\vec{r}, t) = \mu\epsilon \frac{\partial^2 \vec{E}(\vec{r}, t)}{\partial t^2} + \mu\sigma_c \frac{\partial \vec{E}(\vec{r}, t)}{\partial t}} & \underline{\text{and:}} \quad \boxed{\nabla^2 \vec{B}(\vec{r}, t) = \mu\epsilon \frac{\partial^2 \vec{B}(\vec{r}, t)}{\partial t^2} + \mu\sigma_c \frac{\partial \vec{B}(\vec{r}, t)}{\partial t}}
 \end{array}$$

Note that these 3-D wave equations for  $\vec{E}$  and  $\vec{B}$  in a conductor have an additional term that has a single time derivative – which is analogous to a velocity-dependent damping term, e.g. for a harmonic oscillator.

The general solution(s) to the above wave equations are usually in the form of an oscillatory function \* a damping term (i.e. a decaying exponential) – in the direction of the propagation of the *EM* wave, e.g. complex plane-wave type solutions for  $\vec{E}$  and  $\vec{B}$  associated with the above wave equation(s) are of the general form:

$$\boxed{\tilde{\vec{E}}(z, t) = \tilde{\vec{E}}_o e^{i(\tilde{k}z - \omega t)}} \quad \text{and} \quad \boxed{\tilde{\vec{B}}(z, t) = \tilde{\vec{B}}_o e^{i(\tilde{k}z - \omega t)} = \frac{1}{\omega} \tilde{\vec{k}} \times \tilde{\vec{E}}(z, t)}$$

With {frequency-dependent} complex wave number:  $\boxed{\tilde{k}(\omega) = k(\omega) + i\kappa(\omega)}$  where

$\boxed{k(\omega) = \text{Re}(\tilde{k}(\omega))}$  and  $\boxed{\kappa(\omega) = \text{Im}(\tilde{k}(\omega))}$  and corresponding complex wave vector

$\boxed{\tilde{\vec{k}}(\omega) = \tilde{k}(\omega) \hat{z}}$  (in the  $+\hat{z}$  direction here), i.e.  $\boxed{\tilde{\vec{k}}(\omega) = (k(\omega) + i\kappa(\omega)) \hat{z}}$  and with:

$\boxed{\tilde{k}^2(\omega) = \mu\epsilon\omega^2 + i\mu\sigma_c\omega}$   $\Leftarrow$  plug above expressions for  $\tilde{\vec{E}}(z, t)$  and  $\tilde{\vec{B}}(z, t)$  into the above wave equations to derive/show/prove – please work this out/explicitly do this yourselves !!!

$$\boxed{\tilde{\vec{E}}(z,t) = \tilde{\vec{E}}_o e^{i(\tilde{k}z - \omega t)}} \text{ and } \boxed{\tilde{\vec{B}}(z,t) = \tilde{\vec{B}}_o e^{i(\tilde{k}z - \omega t)}} \text{ with } \boxed{\tilde{k}(\omega) = k(\omega) + i\kappa(\omega)} \text{ and } \boxed{\tilde{k}^2(\omega) = \mu\varepsilon\omega^2 + i\mu\sigma_c\omega}$$

$$\boxed{k(\omega) = \text{Re}(\tilde{k}(\omega)) = \omega \sqrt{\frac{\varepsilon\mu}{2} \left[ \sqrt{1 + \left(\frac{\sigma_c}{\varepsilon\omega}\right)^2} + 1 \right]^{1/2}}} \text{ and } \boxed{\kappa(\omega) = \text{Im}(\tilde{k}(\omega)) = \omega \sqrt{\frac{\varepsilon\mu}{2} \left[ \sqrt{1 + \left(\frac{\sigma_c}{\varepsilon\omega}\right)^2} - 1 \right]^{1/2}}$$

Note that the imaginary part of  $\tilde{k}$ ,  $\kappa = \text{Im}(\tilde{k})$  results in an exponential attenuation/damping of the monochromatic plane *EM* wave with increasing  $z$ :

$$\boxed{\tilde{\vec{E}}(z,t) = \tilde{\vec{E}}_o e^{-\kappa z} e^{i(kz - \omega t)}} \text{ and: } \boxed{\tilde{\vec{B}}(z,t) = \tilde{\vec{B}}_o e^{-\kappa z} e^{i(kz - \omega t)} = \frac{1}{\omega} \tilde{k} \times \tilde{\vec{E}}(z,t) = \frac{1}{\omega} \tilde{k} \times \tilde{\vec{E}}_o e^{-\kappa z} e^{i(kz - \omega t)}}$$

n.b. these solutions satisfy the above wave equations for any choice  $\tilde{\vec{E}}_o$  and  $\tilde{\vec{B}}_o$

The characteristic distance over which  $\vec{E}$  and  $\vec{B}$  are attenuated/reduced to  $1/e = e^{-1} = 0.3679$  of their initial values (at  $z = 0$ ) is known as the skin depth,  $\boxed{\delta_{sc}(\omega) \equiv 1/\kappa(\omega)}$  (SI units: meters).

$$\text{i.e. } \boxed{\delta_{sc}(\omega) = \frac{1}{\kappa(\omega)} = \frac{1}{\omega \sqrt{\frac{\varepsilon\mu}{2} \left[ \sqrt{1 + \left(\frac{\sigma_c}{\varepsilon\omega}\right)^2} - 1 \right]^{1/2}}} \Rightarrow \begin{cases} \tilde{\vec{E}}(z = \delta_{sc}, t) = \tilde{\vec{E}}_o e^{-1} e^{i(kz - \omega t)} \\ \tilde{\vec{B}}(z = \delta_{sc}, t) = \tilde{\vec{B}}_o e^{-1} e^{i(kz - \omega t)} \end{cases}$$

The real part of  $\tilde{k}$   $\boxed{k(\omega) = \text{Re}(\tilde{k}(\omega))}$  determines the spatial wavelength  $\lambda(\omega)$ , the propagation speed  $v(\omega)$  of the monochromatic plane *EM* wave in the conductor, and also the index of refraction:

$$\boxed{\lambda(\omega) = \frac{2\pi}{k(\omega)} = \frac{2\pi}{\text{Re}(\tilde{k}(\omega))}}, \quad \boxed{v(\omega) = \frac{\omega}{k(\omega)} = \frac{\omega}{\text{Re}(\tilde{k}(\omega))}} \text{ and } \boxed{n(\omega) = \frac{c}{v(\omega)} = \frac{ck(\omega)}{\omega} = \frac{c \text{Re}(\tilde{k}(\omega))}{\omega}}$$

The above plane wave solutions satisfy the above wave equations(s) for any choice of  $\tilde{\vec{E}}_o$  and  $\tilde{\vec{B}}_o$ . As we have seen before, it can also be shown here that Maxwell's equations 1) and 2) ( $\vec{\nabla} \cdot \vec{E} = 0$  and  $\vec{\nabla} \cdot \vec{B} = 0$ ) rule out any {longitudinal}  $z$ -components for  $\vec{E}$  and  $\vec{B}$  (for *EM* waves propagating in the  $+\hat{z}$ -direction)  $\Rightarrow \vec{E}$  and  $\vec{B}$  are purely transverse waves (as before), even in a conductor!

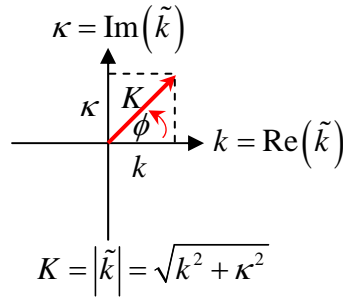
If we consider e.g. a linearly polarized monochromatic plane *EM* wave propagating in the  $+\hat{z}$ -direction in a conducting medium, e.g.  $\boxed{\tilde{\vec{E}}(z,t) = \tilde{\vec{E}}_o e^{-\kappa z} e^{i(kz - \omega t)} \hat{x}}$ , then:

$$\boxed{\tilde{\vec{B}}(z,t) = \frac{1}{\omega} \tilde{k} \times \tilde{\vec{E}}(z,t) = \left( \frac{\tilde{k}}{\omega} \right) \tilde{\vec{E}}_o e^{-\kappa z} e^{i(kz - \omega t)} \hat{y}}$$

$$\Rightarrow \tilde{\vec{E}}(z,t) \perp \tilde{\vec{B}}(z,t) \perp \hat{z} \quad (+\hat{z} = \text{propagation direction})$$

The complex wavenumber  $\tilde{k} = k + ik = Ke^{i\phi}$  where:  $K \equiv |\tilde{k}| = \sqrt{k^2 + \kappa^2}$  and  $\phi \equiv \tan^{-1}\left(\frac{\kappa}{k}\right)$

In the complex plane:



Then we see that:  $\tilde{\vec{E}}(z, t) = \tilde{E}_o e^{-\kappa z} e^{i(kz - \omega t)} \hat{x}$  has  $\tilde{E}_o = E_o e^{i\delta_E}$   $\tilde{k} = Ke^{i\phi}$

and that:  $\tilde{\vec{B}}(z, t) = \tilde{B}_o e^{-\kappa z} e^{i(kz - \omega t)} \hat{y} = \frac{\tilde{k}}{\omega} \tilde{E}_o e^{-\kappa z} e^{i(kz - \omega t)} \hat{y}$  has  $\tilde{B}_o = B_o e^{i\delta_B} = \frac{\tilde{k}}{\omega} \tilde{E}_o = \frac{Ke^{i\phi}}{\omega} \tilde{E}_o = \frac{Ke^{i\phi}}{\omega} E_o e^{i\delta_E}$

Thus, we see that:  $B_o e^{i\delta_B} = \frac{Ke^{i\phi}}{\omega} E_o e^{i\delta_E} = \frac{K}{\omega} E_o e^{i(\delta_E + \phi)} = \frac{\sqrt{k^2 + \kappa^2}}{\omega} E_o e^{i(\delta_E + \phi)}$

i.e., we see that in a conductor,  $\vec{E}$  and  $\vec{B}$  are no longer in phase with each other!!!

Phases of  $\vec{E}$  and  $\vec{B}$ :  $\delta_B = \delta_E + \phi$  or:  $\delta_B - \delta_E = \phi \Leftarrow$  magnetic field lags behind electric field!!!

We also see that:  $\frac{B_o}{E_o} = \frac{K}{\omega} = \left[ \varepsilon\mu \sqrt{1 + \left(\frac{\sigma_c}{\varepsilon\omega}\right)^2} \right]^{1/2} \neq \frac{1}{c}$

The real/physical  $\vec{E}$  and  $\vec{B}$  fields associated with linearly polarized monochromatic plane EM waves propagating in a conducting medium are exponentially damped:

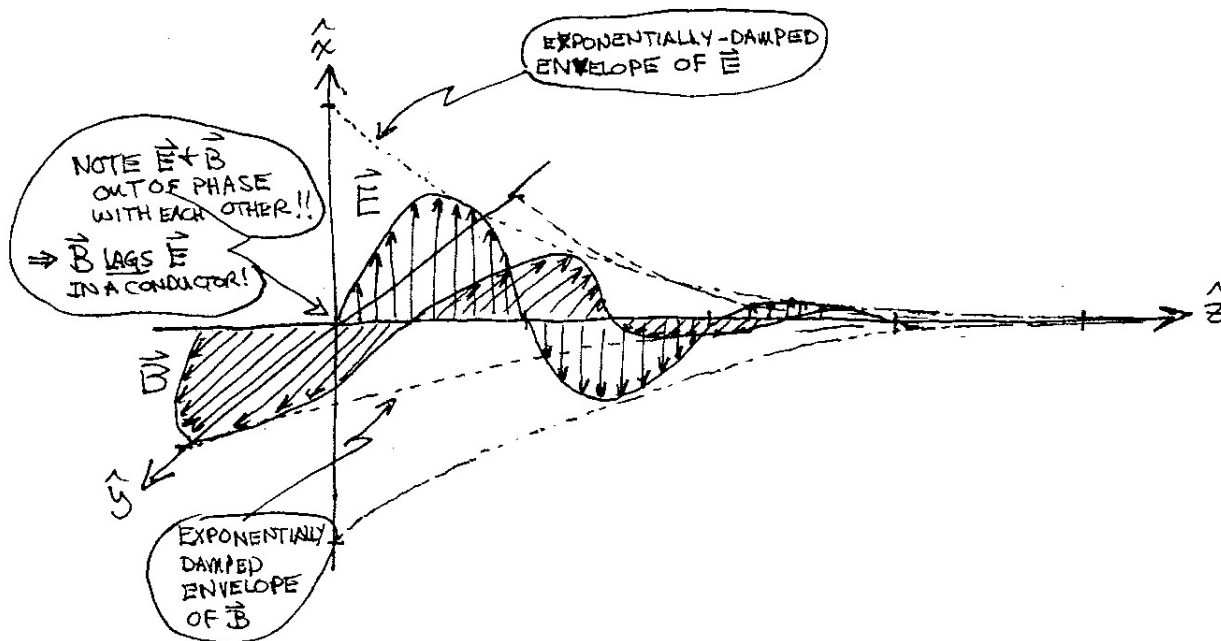
$$\begin{aligned} \vec{E}(z, t) &= \text{Re}\left(\tilde{\vec{E}}(z, t)\right) = E_o e^{-\kappa z} \cos(kz - \omega t + \delta_E) \hat{x} & \delta_B = \delta_E + \phi \\ \vec{B}(z, t) &= \text{Re}\left(\tilde{\vec{B}}(z, t)\right) = B_o e^{-\kappa z} \cos(kz - \omega t + \{\delta_E + \phi\}) \hat{y} = B_o e^{-\kappa z} \cos(kz - \omega t + \delta_B) \hat{y} \end{aligned}$$

$$\begin{aligned} \frac{B_o}{E_o} = \frac{K(\omega)}{\omega} &= \left[ \varepsilon\mu \sqrt{1 + \left(\frac{\sigma_c}{\varepsilon\omega}\right)^2} \right]^{1/2} \text{ where } K(\omega) \equiv |\tilde{k}(\omega)| = \sqrt{k^2(\omega) + \kappa^2(\omega)} = \omega \left[ \varepsilon\mu \sqrt{1 + \left(\frac{\sigma_c}{\varepsilon\omega}\right)^2} \right]^{1/2} \\ \text{and } \delta_B = \delta_E + \phi, \phi &\equiv \tan^{-1}\left(\frac{\kappa(\omega)}{k(\omega)}\right) \text{ and } \tilde{k}(\omega) = (k(\omega) + i\kappa(\omega)) \hat{z}, \tilde{k}(\omega) = |\tilde{k}(\omega)| = k(\omega) + i\kappa(\omega) \end{aligned}$$

The skin depth in a conductor:

$$\delta_{sc}(\omega) \equiv 1/\kappa(\omega) = \frac{1}{\omega \sqrt{\frac{\epsilon\mu}{2}} \left[ \sqrt{1 + \left(\frac{\sigma_c}{\epsilon\omega}\right)^2} - 1 \right]^{1/2}}$$

Distance over which the  $\vec{E}$  and  $\vec{B}$  fields fall to  $1/e = e^{-1} = 0.3679$  of their initial values.



Special/Limiting Cases:

a) Good conductors:  $\sigma_c \gg \epsilon\omega$  Conductivity of good conductor  $\sigma_c \rightarrow \infty$  (i.e.  $\rho_c = 1/\sigma_c \rightarrow 0$ ).

Since  $\tilde{k} = k + ik$  and  $\sigma_c \gg \epsilon\omega$ , i.e.  $\left(\frac{\sigma_c}{\epsilon\omega} \gg 1\right)$  then:

$$k \equiv \omega \sqrt{\frac{\epsilon\mu}{2}} \left[ \sqrt{1 + \left(\frac{\sigma_c}{\epsilon\omega}\right)^2} + 1 \right]^{1/2} \approx \omega \sqrt{\frac{\epsilon\mu}{2}} \left[ \sqrt{1 + \left(\frac{\sigma_c}{\epsilon\omega}\right)^2} \right]^{1/2} \approx \omega \sqrt{\frac{\epsilon\mu}{2}} \left[ \frac{\sigma_c}{\epsilon\omega} \right]^{1/2} = \omega \sqrt{\frac{\epsilon\mu\sigma_c}{2\epsilon\omega}} = \sqrt{\frac{\omega\mu\sigma_c}{2}}$$

and:

$$\kappa \equiv \omega \sqrt{\frac{\epsilon\mu}{2}} \left[ \sqrt{1 + \left(\frac{\sigma_c}{\epsilon\omega}\right)^2} - 1 \right]^{1/2} \approx \omega \sqrt{\frac{\epsilon\mu}{2}} \left[ \sqrt{1 + \left(\frac{\sigma_c}{\epsilon\omega}\right)^2} \right]^{1/2} \approx \omega \sqrt{\frac{\epsilon\mu}{2}} \left[ \frac{\sigma_c}{\epsilon\omega} \right]^{1/2} = \omega \sqrt{\frac{\epsilon\mu\sigma_c}{2\epsilon\omega}} = \sqrt{\frac{\omega\mu\sigma_c}{2}}$$

$\therefore$  In a good conductor  $\sigma_c \gg \epsilon\omega$ :  $k(\omega) \approx \kappa(\omega) \approx \sqrt{\frac{\omega\mu\sigma_c}{2}}$  and skin depth,  $\delta_{sc} \equiv \frac{1}{\kappa} \approx \sqrt{\frac{2}{\omega\mu\sigma_c}}$ .

Wavenumber,  $k(\omega) \equiv 2\pi/\lambda(\omega) \Rightarrow 2\pi/k(\omega) \approx 2\pi/\kappa(\omega) = 2\pi\delta_{sc}(\omega)$

**FORMULAS FOR EM WAVE PROPAGATION IN A GOOD CONDUCTOR**

$$k(\omega) \approx \kappa(\omega) \approx \sqrt{\frac{\omega\mu\sigma_c}{2}}$$

and:

$$\delta_{sc}(\omega) = \text{skin depth} \equiv \frac{1}{\kappa(\omega)} \approx \sqrt{\frac{2}{\omega\mu\sigma_c}}$$

$$\text{Wavenumber, } k(\omega) \equiv \frac{2\pi}{\lambda(\omega)} \Rightarrow \lambda(\omega) = \frac{2\pi}{k(\omega)} \approx \frac{2\pi}{\kappa(\omega)} = 2\pi\delta_{sc}(\omega) = 2\pi\sqrt{\frac{2}{\omega\mu\sigma_c}}$$

 n.b. in a *perfect* conductor:  $\sigma_c = \infty$ 

$$\Rightarrow k(\omega) \approx \kappa(\omega) = \sqrt{\frac{\omega\mu\sigma_c}{2}} = \infty$$

$$\Rightarrow \lambda(\omega) = \frac{2\pi}{k(\omega)} = 0$$

$$\delta_{sc}(\omega) = \frac{1}{\kappa(\omega)} \approx \sqrt{\frac{2}{\omega\mu\sigma_c}} = 0$$

$$\phi \equiv (\delta_B - \delta_E) \equiv \tan^{-1}\left(\frac{\kappa(\omega)}{k(\omega)}\right) \approx \tan^{-1}(1)$$

$$\text{But: } \tan^{-1}(1) = 45^\circ = \frac{\pi}{4}$$

$$\Rightarrow \phi = \delta_B - \delta_E = 45^\circ = \frac{\pi}{4}$$

 $\Rightarrow \vec{B}$  lags  $\vec{E}$  by  $\approx 45^\circ$  in a *good* conductor.

 n.b. In a *perfect* conductor:  $\sigma_c = \infty, \phi \equiv 45^\circ = \frac{\pi}{4}$ 
In a typical good conductor (e.g. gold/silver/copper/...):  $(\sigma_c/\varepsilon\omega) \gg 1$ 

 For optical frequencies/visible light region:  $\omega \approx 10^{16}$  radians/sec. A good conductor typically has  $\sigma_c \approx 10^7$  Siemens/m and  $\varepsilon \approx 3\varepsilon_0$ , and at optical frequencies:  $(\sigma_c/\varepsilon\omega) \approx 37.7 \gg 1$  is satisfied.

If the conductor is non-magnetic (e.g. copper, aluminum, gold, silver, platinum... etc.)

$$\Rightarrow \mu \approx \mu_0 = 4\pi \times 10^{-7} \text{ Henrys/m.}$$

$$\text{Then: } k(\omega) \approx \kappa(\omega) \approx \sqrt{\frac{\omega\mu\sigma_c}{2}} \approx \sqrt{\frac{\omega\mu_0\sigma_c}{2}} = \left[ \frac{10^{16} \times 4\pi \times 10^{-7} \times 10^7}{2} \right]^{1/2} \approx 2.51 \times 10^8 \text{ radians/m}$$

$$\text{And: } \lambda(\omega) = 2\pi/k(\omega) = \text{wavelength in good conductor} \approx 2.51 \times 10^{-8} \text{ m} = 25.1 \text{ nm}$$

$$\text{cf w/ vacuum wavelength: } \lambda_o = \frac{2\pi}{k_o} = \frac{2\pi c}{\omega} = \frac{c}{f} \approx \frac{2\pi \times 3 \times 10^8}{10^{16}} = 1.885 \times 10^{-7} \text{ m} = 188.5 \text{ nm}$$

$$\Rightarrow \lambda(\omega) \approx 25.1 \text{ nm} \left( \text{good conductor} \right) \ll \lambda_o = 188.5 \text{ nm} \left( \text{vacuum wavelength} \right)$$

$$\text{Vacuum/conductor } \lambda\text{-ratio: } \left( \frac{\lambda_o}{\lambda(\omega)} \right) = \frac{188.5 \text{ nm}}{25.1 \text{ nm}} \approx 7.52 \text{ at optical frequencies, } \omega \approx 10^{16} \text{ rad/sec.}$$

$$\text{Skin depth: } \delta_{sc}(\omega) = \frac{1}{\kappa(\omega)} \approx \frac{\lambda(\omega)}{2\pi} \approx 4.0 \times 10^{-9} \text{ m} = 4.0 \text{ nm} \text{ !!!}$$

 $\Rightarrow$  This explains why metals are opaque {and also explains why silvered sunglasses work} at optical frequencies,  $\omega \approx 10^{16}$  radians/sec.

Compare these results for *EM* waves propagating in conductors at optical frequencies to those for *EM* waves propagating in conductors, but with very low frequencies – e.g. the AC line frequency,  $f_{AC} = 60 \text{ Hz} \Rightarrow \omega_{AC} = 2\pi f_{AC} = 120\pi \text{ rad/sec}$ , where the criterion for a good conductor,  $\left(\frac{\sigma_C}{\epsilon\omega}\right) \approx 10^{15} \gg 1$  is certainly well-satisfied:

$$\text{At } f = 60\text{Hz: } \begin{cases} k_{AC} \approx \kappa_{AC} = \sqrt{\frac{\omega\mu\sigma_C}{2}} = \left[ \frac{120\pi \times 4\pi \times 10^{-7} \times 10^7}{2} \right] = 48.7 \text{ radians/m} \\ \lambda_{AC} = \frac{2\pi}{k} = 0.129 \text{ m} = 12.9 \text{ cm}, \lambda_{oAC} = 5 \times 10^6 \text{ m!!} \\ \frac{\lambda_{oAC}}{\lambda_{AC}} = \frac{5 \times 10^6 \text{ m}}{0.129 \text{ m}} \approx 3.87 \times 10^7 \text{ !!} \\ 60 \text{ Hz AC skin depth: } \delta_{sc}^{AC} = \frac{\lambda_{AC}}{2\pi} \approx 2.05 \times 10^{-2} \text{ m} = 2.05 \text{ cm!!} \end{cases}$$

$\Rightarrow$  Need at least  $3-4 \times \delta_{sc} \approx \text{several} \rightarrow 10 \text{ cm}$  to screen out unwanted 60 Hz AC signals !!

EM Wave Energy Densities in a Good Conductor:  $\left(\frac{\sigma_C}{\epsilon\omega}\right) \gg 1$

$$u_{EM} = u_E^{EM} + u_M^{EM} = \left(\frac{1}{2} \epsilon E^2\right) + \left(\frac{1}{2\mu} B^2\right) = \left(\frac{1}{2} \epsilon \vec{E} \cdot \vec{E}\right) + \left(\frac{1}{2\mu} \vec{B} \cdot \vec{B}\right)$$

$$\phi \equiv (\delta_B - \delta_E) \approx \frac{\pi}{4} = 45^\circ \text{ \{in a good conductor\}}$$

$$\vec{E}(z,t) = E_o e^{-\kappa z} \cos(kz - \omega t + \delta_E) \hat{x} \quad \text{and} \quad \vec{B}(z,t) = B_o e^{-\kappa z} \cos(kz - \omega t + \delta_E + \phi) \hat{y}$$

Where:  $B_o = \frac{K(\omega)}{\omega} E_o = \left[ \epsilon\mu \sqrt{1 + \left(\frac{\sigma_C}{\epsilon\omega}\right)^2} \right]^{1/2} E_o \approx \sqrt{\frac{\mu\sigma_C}{\omega}} E_o = \sqrt{\frac{\mu\sigma_C}{\omega}} E_o$  for a good conductor,

And:  $k(\omega) \approx \kappa(\omega) \approx \sqrt{\frac{\omega\mu\sigma_C}{2}}$ ,

$$v(\omega) = \frac{\omega}{k(\omega)} \approx \frac{\omega}{\sqrt{\frac{\omega\mu\sigma_C}{2}}} = \sqrt{\frac{2\omega}{\omega\sigma_C}} = \frac{c}{n(\omega)} \text{ for a good conductor.}$$

Then:

$$u_E^{EM}(z,t) = \frac{1}{2} \epsilon E^2 = \frac{1}{2} \epsilon \vec{E} \cdot \vec{E} = \frac{1}{2} \epsilon E_o^2 e^{-2\kappa z} \cos^2(kz - \omega t + \delta_E) \quad \text{and:}$$

$$u_M^{EM}(z,t) = \frac{1}{2\mu} B^2 = \frac{1}{2\mu} \vec{B} \cdot \vec{B} = \frac{1}{2\mu} B_o^2 e^{-2\kappa z} \cos^2(kz - \omega t + \delta_E + \phi) = \frac{\sigma_C}{2\omega} E_o^2 e^{-2\kappa z} \cos^2(kz - \omega t + \delta_E + \phi)$$

Time-averaging these quantities over one complete cycle:  $\langle u(z,t) \rangle \equiv \frac{1}{\tau} \int_0^\tau u(z,t) dt$

$$\langle u_E^{EM}(z,t) \rangle = \frac{1}{2} \varepsilon E_o^2 e^{-2\kappa z} \underbrace{\frac{1}{\tau} \int_0^\tau \cos^2(kz - \omega t + \delta_E) d\tau}_{=\frac{1}{2}} = \frac{1}{4} \varepsilon E_o^2 e^{-2\kappa z}$$

$$\langle u_M^{EM}(z,t) \rangle = \frac{\sigma_C}{2\omega} E_o^2 e^{-2\kappa z} \underbrace{\frac{1}{\tau} \int_0^\tau \cos^2(kz - \omega t + \delta_E + \phi) d\tau}_{=\frac{1}{2}} = \frac{1}{4} \left( \frac{\sigma_C}{\omega} \right) E_o^2 e^{-2\kappa z}$$

$$\therefore \langle u_{Tot}^{EM}(z,t) \rangle = \langle u_E^{EM}(z,t) \rangle + \langle u_M^{EM}(z,t) \rangle = \frac{1}{4} \varepsilon \left( 1 + \frac{\sigma_C}{\varepsilon\omega} \right) E_o^2 e^{-2\kappa z} \quad \text{n.b. Exponentially attenuated in } z \text{ !!!}$$

**But:**  $\left( \frac{\sigma_C}{\varepsilon\omega} \right) \gg 1$  for a good conductor,  $\Rightarrow \langle u_{Tot}^{EM}(z,t) \rangle \approx \frac{1}{2} \left( \frac{\sigma_C}{\varepsilon\omega} \right) \left[ \frac{1}{2} \varepsilon E_o^2 e^{-2\kappa z} \right]$

i.e. the ratio:  $\frac{\langle u_M^{EM}(z,t) \rangle}{\langle u_E^{EM}(z,t) \rangle} = \left( \frac{\sigma_C}{\varepsilon\omega} \right) \gg 1$  or  $\langle u_M^{EM}(z,t) \rangle \gg \langle u_E^{EM}(z,t) \rangle$  for a good conductor.

$\Rightarrow$  Vast majority of *EM* wave energy is carried by the magnetic field in a good conductor !!!

Poynting's Vector:  $\vec{S} = \frac{1}{\mu} \vec{E} \times \vec{B} \Rightarrow \langle \vec{S}(z,t) \rangle = \frac{1}{\mu} \langle \vec{E} \times \vec{B} \rangle = \frac{1}{2\mu} E_o B_o e^{-2\kappa z} \cos \phi \hat{z} \leftarrow \phi = \frac{\pi}{4}$

EM wave intensity (aka irradiance):  $I(z) = \langle |\vec{S}(z,t)| \rangle = \frac{1}{2\mu} E_o B_o e^{-2\kappa z} \cos \phi = \frac{1}{2\mu} E_o^2 e^{-2\kappa z} \left( \frac{K}{\omega} \cos \phi \right)$

**But:**  $\frac{K \cos \phi}{\omega} = \frac{k}{\omega} \approx \sqrt{\frac{\omega\mu\sigma_C}{2}} = \sqrt{\frac{\mu\sigma_C}{2\omega}} \therefore I(z) = \langle |\vec{S}(z,t)| \rangle = \frac{1}{2\mu} \left( \frac{k}{\omega} \right) E_o^2 e^{-2\kappa z} = \frac{1}{2} \sqrt{\frac{\sigma_C}{2\mu\omega}} E_o^2 e^{-2\kappa z}$

b) Special/Limiting Case of a Fair Conductor:  $\sigma_C \approx \varepsilon\omega \Rightarrow$  Must use exact formulae!

c) Special/Limiting Case of a Poor Conductor (i.e. an insulator):

Here:  $\sigma_C \ll \varepsilon\omega$ , i.e.  $\left( \frac{\sigma_C}{\varepsilon\omega} \right) \ll 1$ . Conductivity of poor conductor:  $\sigma_C \rightarrow 0$  (i.e.  $\rho_C = 1/\sigma_C \rightarrow \infty$ ).

Complex wavenumber:  $\tilde{k} = k + ik$ , with  $k = \text{Re}(\tilde{k})$  and  $\kappa = \text{Im}(\tilde{k})$ .

$$k(\omega) \equiv \omega \sqrt{\frac{\varepsilon\mu}{2}} \left[ \sqrt{1 + \left( \frac{\sigma_C}{\varepsilon\omega} \right)^2} + 1 \right]^{1/2} \approx \omega \sqrt{\frac{\varepsilon\mu}{2}} \left[ 1 + \frac{1}{2} \left( \frac{\sigma_C}{\varepsilon\omega} \right)^2 + 1 \right]^{1/2} = \omega \sqrt{\frac{\varepsilon\mu}{2}} \left[ 2 + \frac{1}{2} \left( \frac{\sigma_C}{\varepsilon\omega} \right)^2 \right]^{1/2} \approx \omega \sqrt{\varepsilon\mu}$$

$\therefore k(\omega) \approx \omega \sqrt{\varepsilon\mu}$  for a poor conductor.

Likewise:

$$\kappa(\omega) \equiv \omega \sqrt{\frac{\epsilon\mu}{2}} \left[ \sqrt{1 + \left(\frac{\sigma_c}{\epsilon\omega}\right)^2} - 1 \right]^{1/2} \approx \omega \sqrt{\frac{\epsilon\mu}{2}} \left[ \lambda + \frac{1}{2} \left(\frac{\sigma_c}{\epsilon\omega}\right)^2 - \lambda \right]^{1/2} = \omega \sqrt{\frac{\epsilon\mu\sigma_c^2}{4\epsilon^2\omega^2}} \approx \frac{1}{2} \sigma_c \sqrt{\frac{\mu}{\epsilon}}$$

$$\therefore \kappa(\omega) \approx \frac{1}{2} \sigma_c \sqrt{\frac{\mu}{\epsilon}} \text{ for a } \underline{\text{poor}} \text{ conductor.}$$

In a poor conductor  $\left(\frac{\sigma_c}{\epsilon\omega}\right) \ll 1$ , the ratio:  $\left(\frac{\kappa(\omega)}{k(\omega)}\right) \approx \frac{\frac{1}{2} \sigma_c \sqrt{\frac{\mu}{\epsilon}}}{\omega \sqrt{\epsilon\mu}} = \frac{1}{2} \left(\frac{\sigma_c}{\epsilon\omega}\right) \ll 1$  i.e.  $\kappa(\omega) \ll k(\omega)$ .

$\Rightarrow$  Complex wavenumber  $\tilde{k} \equiv k + ik$  is primarily real, because  $\kappa \ll k$  in a poor conductor.

Phase angle in a poor conductor:  $\phi \equiv \delta_B - \delta_E = \tan^{-1}\left(\frac{\kappa(\omega)}{k(\omega)}\right) = \tan^{-1}\left(\frac{1}{2}\left(\frac{\sigma_c}{\epsilon\omega}\right)\right) \approx \frac{1}{2}\left(\frac{\sigma_c}{\epsilon\omega}\right) \ll 1$

$\Rightarrow \delta_B = \delta_E + \phi \approx \delta_E$ , i.e.  $\vec{B}$  and  $\vec{E}$  are nearly in phase with each other in a poor conductor.

In a typical poor conductor, e.g. pure water:

Water has a huge static electric permittivity (due to permanent electric dipole moment of water molecule):  $\epsilon_{H_2O} \approx 81\epsilon_o$  (at zero Hz, i.e.  $f = 0$ ) (at  $P = 1 \text{ ATM}$  and  $T = 20^\circ \text{ C}$ ), however, at optical frequencies ( $\omega \approx 10^{16} \text{ rad/sec}$ ):  $\epsilon_{H_2O}(\omega) \approx 1.777\epsilon_o$ , where  $\epsilon_o = 8.85 \times 10^{-12} \text{ Farads/m}$ .

Since water is non-magnetic:  $\mu_{H_2O} \approx \mu_o = 4\pi \times 10^{-7} \text{ Henrys/m}$

$\Rightarrow$  index of refraction:  $n_{H_2O}(\omega) = \sqrt{\epsilon_{H_2O}(\omega)\mu_{H_2O}/\epsilon_o\mu_o} \approx 1.333$  at optical frequencies.

The conductivity of pure water is:  $\sigma_c^{H_2O} = 1/\rho_c^{H_2O} \approx 1/2.5 \times 10^5 \Omega\text{-m} = 4.0 \times 10^{-6} \text{ Siemens/m}$

(at  $P = 1 \text{ ATM}$  and  $T = 20^\circ \text{ C}$ ). Thus, the criteria for a poor conductor  $(\sigma_c/\epsilon\omega) \approx 2.54 \times 10^{-11} \ll 1$  is certainly satisfied at optical frequencies.

The wavenumber in pure  $H_2O$  at optical frequencies is:

$$k_{H_2O}(\omega) \approx \omega \sqrt{\epsilon\mu} \approx \omega \sqrt{\epsilon\mu_o} = 10^{16} \sqrt{1.777 \times 8.85 \times 4\pi \times 10^{-7}} \approx 4.45 \times 10^7 \text{ radians/m}$$

The wavelength in pure  $H_2O$  is:  $\lambda_{H_2O} = 2\pi/k_{H_2O} = 1.413 \times 10^{-7} \text{ m} = 141.3 \text{ nm}$  at optical frequencies.

cf w/ the vacuum wavelength:  $\lambda_o = c/f = 2\pi c/\omega = 1.885 \times 10^{-7} \text{ m} = 188.5 \text{ nm}$

Note that the optical wavelength ratio:  $\left(\frac{\lambda_o}{\lambda_{H_2O}}\right) = \frac{188.5 \text{ nm}}{141.3 \text{ nm}} = 1.333 = n_{H_2O}$ , since  $\lambda_{H_2O} = \lambda_o/n_{H_2O}$

in a poor conductor!!!

Skin depth:  $\delta_{sc}(\omega) \equiv \frac{1}{\kappa(\omega)} \approx \frac{1}{\frac{1}{2}\sigma_C\sqrt{\mu/\epsilon}}$  for a poor conductor  $\left(\frac{\sigma_C}{\epsilon\omega}\right) \ll 1$ .

For pure  $H_2O$  at optical frequencies:

$$\kappa_{H_2O}(\omega) \approx \frac{1}{2}\sigma_C\sqrt{\frac{\mu}{\epsilon}} \approx \frac{1}{2}\sigma_C\sqrt{\frac{\mu_o}{\epsilon}} = \frac{1}{2}\left(\frac{1}{2.5 \times 10^5}\right)\sqrt{\frac{4\pi \times 10^{-7}}{1.777 \times 8.85 \times 10^{-12}}} \approx 5.65 \times 10^{-4} \text{ rad/m}$$

$\delta_{sc}^{H_2O}(\omega) \equiv \frac{1}{\kappa_{H_2O}} = 1.7688 \times 10^3 \text{ m} = 1.77 \text{ km}$	n.b. neglects/ignores <u>Rayleigh scattering</u> process – visible light photons <u>elastically</u> scattering off of $H_2O$ molecules. $\lambda_{atten}^{vis} \approx 10 \text{ m}$
--	--

Ratio:  $\left(\frac{\kappa_{H_2O}(\omega)}{k_{H_2O}(\omega)}\right) = \frac{\frac{1}{2}\sigma_C\sqrt{\frac{\mu}{\epsilon}}}{\omega\sqrt{\epsilon\mu}} = \frac{1}{2}\left(\frac{\sigma_C}{\epsilon\omega}\right) = \frac{1}{2}\left(\frac{1}{2.5 \times 10^5}\right)\frac{1}{1.777 \times 8.85 \times 10^{-12} \times 10^{16}} = 1.27 \times 10^{-11} \ll 1$

Phase difference:  $\phi \equiv \delta_B - \delta_E = \tan^{-1}\left(\frac{\kappa_{H_2O}}{k_{H_2O}}\right) \approx 1.27 \times 10^{-11} \text{ radians } (\ll 1)$  i.e.  $\delta_B = \delta_E + \phi \approx \delta_E$

$\Rightarrow \vec{B}$  and  $\vec{E}$  are nearly in phase with each other in pure  $H_2O$  at optical frequencies.

For pure  $H_2O$  at low frequency – e.g. 60 Hz AC line frequency ( $\omega_{AC} = 2\pi f_{AC} = 120\pi \text{ rad/sec}$ ):

The electric permittivity at  $f = 60 \text{ Hz}$  is  $\epsilon_{H_2O}^{AC}(f \approx 60 \text{ Hz}) \approx 80\epsilon_o = 80 \times 8.85 \times 10^{-12} \text{ Farads/m}$

and  $\mu_{H_2O}^{AC} \approx \mu_o = 4\pi \times 10^{-7} \text{ Henrys/m}$ . Conductivity of pure  $H_2O$ :  $\sigma_C^{H_2O} = 4.0 \times 10^{-6} \text{ Siemens/m}$

Note that the criteria for a poor conductor:  $\left(\frac{\sigma_C}{\epsilon_{H_2O}^{AC}\omega_{AC}}\right) \approx \frac{4.0 \times 10^{-6}}{8 \times 8.85 \times 10^{-12} \cdot 120\pi} = 149.9 \ll 1$

is not satisfied at the 60 Hz AC line frequency – i.e. at low enough frequencies, even poor conductors such as pure water are actually quite good conductors !!!

Thus, for the following, we must use the good conductor approximations:

$$k_{AC}^{H_2O}(\omega) \approx \kappa_{AC}^{H_2O}(\omega) \approx \sqrt{\frac{\omega_{AC}\mu_{AC}^{H_2O}\sigma_C}{2}} \approx \sqrt{\frac{\omega_{AC}\mu_o\sigma_C}{2}} = \sqrt{\frac{120\pi \cdot 4\pi \times 10^{-7} \cdot 4.0 \times 10^{-6}}{2}} = 3.08 \times 10^{-5} \text{ rads/m}$$

$$\lambda_{AC}^{H_2O}(\omega) \approx \frac{2\pi}{k_{AC}^{H_2O}(\omega)} = 2.04 \times 10^5 \text{ m} \quad \text{cf w/ vacuum wavelength: } \lambda_o = c/f_{AC} = \frac{2\pi c}{\omega_{AC}} = 5.00 \times 10^6 \text{ m}$$

Vacuum/good conductor wavelength ratio:  $\left(\frac{\lambda_o}{\lambda_{AC}^{H_2O}}\right) = \frac{5.00 \times 10^6 \text{ m}}{2.04 \times 10^5 \text{ m}} \approx 24.495$

Skin depth for pure  $H_2O$  at 60 Hz AC line frequency:  $\delta_{H_2O}^{AC} \equiv 1/\kappa_{H_2O}^{AC} \approx 3.25 \times 10^4 \text{ m} = 32.5 \text{ km}$

This may seem like a large distance scale associated with the attenuation of the 60 Hz *EM* waves propagating in pure water, however compare the skin depth to the wavelength at this frequency:  $\delta_{H_2O}^{AC} = 32.5 \text{ km}$  vs.  $\lambda_{AC}^{H_2O} = 1.77 \times 10^6 \text{ m}$ , i.e. we see that  $\delta_{H_2O}^{AC} \ll \lambda_{H_2O}^{AC}$ , as we expect for the case of a good conductor !!!

The ratio  $\left(\kappa_{H_2O}^{AC}/k_{H_2O}^{AC}\right) \approx 1$  for pure  $H_2O$  at 60 Hz AC line frequency, which is what we expect for a good conductor {this ratio should be  $\ll 1$  for a poor conductor}.

Thus, the phase difference is:  $\phi \equiv \delta_B - \delta_E = \tan^{-1}\left(\kappa_{H_2O}^{AC}/k_{H_2O}^{AC}\right) \approx \tan^{-1}(1) = \frac{\pi}{4} = 45^\circ$

which again is what we expect for a good conductor, i.e.  $\vec{B}$  lags  $\vec{E}$  by  $45^\circ$  !

EM energy densities in a poor conductor:  $\left(\frac{\sigma_c}{\epsilon\omega}\right) \ll 1$

$$u_{EM}(z,t) = u_E^{EM}(z,t) + u_M^{EM}(z,t) = \left(\frac{1}{2}\epsilon E^2\right) + \left(\frac{1}{2\mu}B^2\right) = \left(\frac{1}{2}\epsilon \vec{E} \cdot \vec{E}\right) + \left(\frac{1}{2\mu} \vec{B} \cdot \vec{B}\right)$$

The physical  $\vec{E}$  and  $\vec{B}$  fields are:

$$\vec{E}(z,t) = E_o e^{-\kappa z} \cos(kz - \omega t + \delta_E) \hat{x} \quad \text{and} \quad \vec{B}(z,t) = B_o e^{-\kappa z} \cos(kz - \omega t + \delta_E + \phi) \hat{y}$$

where:  $B_o = \frac{K}{\omega} E_o = \left[\epsilon\mu \sqrt{1 + \left(\frac{\sigma_c}{\epsilon\omega}\right)^2}\right]^{1/2} E_o \approx \sqrt{\epsilon\mu} E_o$  for a poor conductor,  $\left(\frac{\sigma_c}{\epsilon\omega}\right) \ll 1$ .

$k \approx \omega \sqrt{\epsilon\mu} = \frac{\omega}{v}$  where:  $v = \frac{1}{\sqrt{\epsilon\mu}} = c/n$  and:  $n = \sqrt{\frac{\epsilon\mu}{\epsilon_o\mu_o}}$  for a poor conductor.

and:  $\kappa \approx \frac{1}{2}\sigma_c \sqrt{\frac{\mu}{\epsilon}} \ll k \approx \omega \sqrt{\epsilon\mu}$ ,  $K \equiv |\tilde{k}| \approx \omega \sqrt{\epsilon\mu}$  for a poor conductor.

Then:  $u_E^{EM}(z,t) = \frac{1}{2}\epsilon E^2 = \frac{1}{2}c \vec{E} \cdot \vec{E} = \frac{1}{2}\epsilon E_o^2 e^{-2\kappa z} \cos^2(kz - \omega t + \delta_E)$  and:

$$u_M^{EM}(z,t) = \frac{1}{2\mu} B^2 = \frac{1}{2\mu} \vec{B} \cdot \vec{B} = \frac{1}{2\mu} B_o^2 e^{-2\kappa z} \cos^2(kz - \omega t + \delta_E + \phi)$$

Time-averaging these quantities:

$$\langle u_E^{EM}(z,t) \rangle = \frac{1}{4}\epsilon E_o^2 e^{-2\kappa z} \quad \text{and} \quad \langle u_M^{EM}(z,t) \rangle = \frac{1}{4\mu} B_o^2 e^{-2\kappa z} \approx \frac{1}{4\mu} (\epsilon\mu) E_o^2 e^{-2\kappa z} = \frac{1}{4}\epsilon E_o^2 e^{-2\kappa z}$$

$$\therefore \langle u_{Tot}^{EM}(z,t) \rangle = \langle u_E^{EM}(z,t) \rangle + \langle u_M^{EM}(z,t) \rangle \approx \frac{1}{4}\epsilon E_o^2 e^{-2\kappa z} + \frac{1}{4}\epsilon E_o^2 e^{-2\kappa z} = \frac{1}{2}\epsilon E_o^2 e^{-2\kappa z}$$

Thus:  $\langle u_{Tot}^{EM}(z,t) \rangle = \frac{1}{2}\epsilon E_o^2 e^{-2\kappa z}$  for a poor conductor  $\left(\frac{\sigma_c}{\epsilon\omega}\right) \ll 1$ .

The ratio of {time-averaged} electric/magnetic energy densities for a poor conductor:

$$\frac{\langle u_E^{EM}(z,t) \rangle}{\langle u_M^{EM}(z,t) \rangle} = \frac{\frac{1}{4} \epsilon E_o^2 e^{-2\kappa z}}{\frac{1}{4} \epsilon E_o^2 e^{-2\kappa z}} = 1$$

$$\phi \equiv \delta_B - \delta_E = \tan^{-1} \left( \frac{\kappa_{H_2O}}{k_{H_2O}} \right) \ll 1$$

$$\kappa_{H_2O} \approx \frac{1}{2} \sigma_c \sqrt{\frac{\mu_o}{\epsilon}} \ll k_{H_2O} \approx \omega \sqrt{\epsilon \mu_o}$$

$\Rightarrow$   $EM$  wave energy is shared  $\approx$  equally by the  $\vec{E}$  and  $\vec{B}$  fields in a poor conductor!

Poynting's Vector for  $EM$  waves propagating in a poor conductor:

$$\vec{S}(z,t) = \frac{1}{\mu} \vec{E}(z,t) \times \vec{B}(z,t) \Rightarrow \langle \vec{S}(z,t) \rangle = \frac{1}{\mu} \langle \vec{E}(z,t) \times \vec{B}(z,t) \rangle \approx \frac{\sqrt{\epsilon \mu_o}}{2 \mu_o} E_o^2 e^{-2\kappa z} \underbrace{\cos \phi}_{\approx 1} \hat{z}$$

$$\therefore \langle \vec{S}(z,t) \rangle \approx \frac{1}{2} \sqrt{\frac{\epsilon}{\mu_o}} E_o^2 e^{-2\kappa z} \hat{z} \text{ for a } \underline{\text{poor}} \text{ conductor.}$$

Intensity of  $EM$  waves propagating in a poor conductor:

$$I(z,t) = \langle \langle \vec{S}(z,t) \rangle \rangle = \frac{1}{2} \sqrt{\frac{\epsilon}{\mu_o}} E_o^2 e^{-2\kappa z}$$

### Reflection of $EM$ Waves at Normal Incidence from a Conducting Surface:

In the presence of free surface charges  $\sigma_{free}$  and/or free surface currents,  $\vec{K}_{free}$  the boundary conditions obtained from (the integral forms of) Maxwell's equations for reflection and refraction at e.g. a dielectric-conductor interface become:

BC 1): (normal  $\vec{D}$  at interface):  $\epsilon_1 E_1^\perp - \epsilon_2 E_2^\perp = \sigma_{free}$

BC 2): (tangential  $\vec{E}$  at interface):  $E_1^\parallel - E_2^\parallel = 0 \Rightarrow E_1^\parallel = E_2^\parallel$

BC 3): (normal  $\vec{B}$  at interface):  $B_1^\perp - B_2^\perp = 0 \Rightarrow B_1^\perp = B_2^\perp$

BC 4): (tangential  $\vec{H}$  at interface):  $\frac{1}{\mu_1} B_1^\parallel - \frac{1}{\mu_2} B_2^\parallel = \vec{K}_{free} \times \hat{n}_{21}$

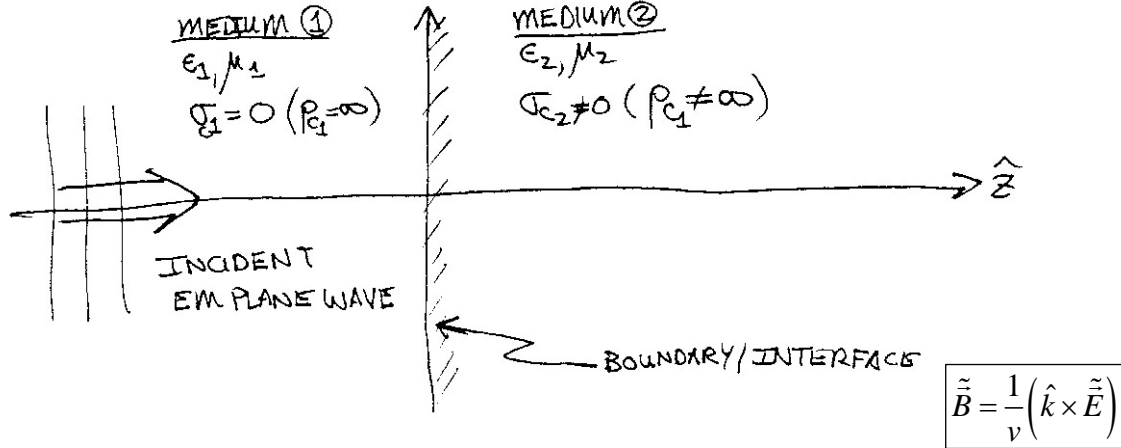
$\perp$  = normal to plane of interface  
 $\parallel$  = parallel to plane of interface

where  $\hat{n}_{21}$  is a unit vector  $\perp$  to the interface, pointing from medium (2) into medium (1).

{n.b. do not confuse  $\hat{n}_{21}$  with the  $EM$  wave polarization vector  $\hat{n}$  !!!}

Note: for Ohmic conductors (i.e. conductors obeying Ohm's Law  $\vec{J}_{free} = \sigma_c \vec{E}$ ) there can be no free surface currents because this would require an infinite  $\vec{E}$ -field at the boundary/interface!

Suppose  $\exists$  a boundary/interface (located in the  $x$ - $y$  plane at  $z = 0$ ) between a non-conducting linear/homogeneous/isotropic medium (1) and a conductor (2). A monochromatic plane EM wave is incident on the interface, that is linearly polarized in  $+\hat{x}$  direction, traveling in the  $+\hat{z}$  direction, approaching the interface/boundary from the left, in medium (1) as shown in the figure below:



$$\tilde{\vec{B}} = \frac{1}{v} (\hat{k} \times \tilde{\vec{E}})$$

Incident EM wave {medium (1)}:  $\tilde{\vec{E}}_{inc}(z, t) = \tilde{E}_{o_{inc}} e^{i(k_1 z - \omega t)} \hat{x}$  and:  $\tilde{\vec{B}}_{inc}(z, t) = \frac{1}{v_1} \tilde{E}_{o_{inc}} e^{i(k_1 z - \omega t)} \hat{y}$

Reflected EM wave {medium (1)}:  $\tilde{\vec{E}}_{refl}(z, t) = \tilde{E}_{o_{refl}} e^{i(-k_1 z - \omega t)} \hat{x}$  and:  $\tilde{\vec{B}}_{refl}(z, t) = -\frac{1}{v_1} \tilde{E}_{o_{refl}} e^{i(-k_1 z - \omega t)} \hat{y}$

Transmitted EM wave {medium (2)}:  $\tilde{\vec{E}}_{trans}(z, t) = \tilde{E}_{o_{trans}} e^{i(\tilde{k}_2 z - \omega t)} \hat{x}$  and:  $\tilde{\vec{B}}_{trans}(z, t) = \frac{\tilde{k}_2}{\omega} \tilde{E}_{o_{trans}} e^{i(\tilde{k}_2 z - \omega t)} \hat{y}$

n.b. complex wavenumber in {conducting} medium (2):  $\tilde{k}_2 = k_2 + i\kappa_2$

In medium (1) EM fields are:  $\tilde{\vec{E}}_{Tot_1}(z, t) = \tilde{\vec{E}}_{inc}(z, t) + \tilde{\vec{E}}_{refl}(z, t)$  and:  $\tilde{\vec{B}}_{Tot_1}(z, t) = \tilde{\vec{B}}_{inc}(z, t) + \tilde{\vec{B}}_{refl}(z, t)$

In medium (2) EM fields are:  $\tilde{\vec{E}}_{Tot_2}(z, t) = \tilde{\vec{E}}_{trans}(z, t)$  and:  $\tilde{\vec{B}}_{Tot_2}(z, t) = \tilde{\vec{B}}_{trans}(z, t)$

Apply BC's at the  $z = 0$  interface in the  $x$ - $y$  plane:

BC 1):  $\epsilon_1 E_1^\perp - \epsilon_2 E_2^\perp = \sigma_{free}$  but:  $E_1^\perp = \tilde{E}_{1z} = 0$  and:  $E_2^\perp = \tilde{E}_{2z} = 0$   $\therefore 0 - 0 = \sigma_{free} \Rightarrow \sigma_{free} = 0$

BC 2):  $E_1^\parallel = E_2^\parallel$   $\therefore \tilde{E}_{o_{inc}} + \tilde{E}_{o_{refl}} = \tilde{E}_{o_{trans}}$

BC 3):  $B_1^\perp = B_2^\perp$  but:  $B_1^\perp = B_{1z} = 0$  and:  $B_2^\perp = B_{2z} = 0 \Rightarrow 0 = 0$

BC 4):  $\frac{1}{\mu_1} B_1^\parallel - \frac{1}{\mu_2} B_2^\parallel = \vec{K}_{free} \times \hat{n}_{21}$  but:  $\vec{K}_{free} = 0$   $\therefore \frac{1}{\mu_1 v_1} (\tilde{E}_{o_{inc}} - \tilde{E}_{o_{refl}}) - \frac{\tilde{k}_2}{\mu_2 \omega} \tilde{E}_{o_{trans}} = 0$

or:  $\tilde{E}_{o_{inc}} - \tilde{E}_{o_{refl}} = \tilde{\beta} \tilde{E}_{o_{trans}}$  with:  $\tilde{\beta} \equiv \left( \frac{\mu_1 v_1 \tilde{k}_2}{\mu_2 \omega} \right) = \left( \frac{\mu_1 v_1}{\mu_2 \omega} \right) \tilde{k}_2$

Thus we obtain: 
$$\tilde{E}_{o_{refl}} = \left( \frac{1-\tilde{\beta}}{1+\tilde{\beta}} \right) \tilde{E}_{o_{inc}} \quad \text{and:} \quad \tilde{E}_{o_{trans}} = \frac{2}{(1+\tilde{\beta})} \tilde{E}_{o_{inc}}$$

or: 
$$\left( \frac{\tilde{E}_{o_{refl}}}{\tilde{E}_{o_{inc}}} \right) = \left( \frac{1-\tilde{\beta}}{1+\tilde{\beta}} \right) \quad \text{and:} \quad \left( \frac{\tilde{E}_{o_{trans}}}{\tilde{E}_{o_{inc}}} \right) = \frac{2}{(1+\tilde{\beta})} \quad \text{with:} \quad \tilde{\beta} \equiv \left( \frac{\mu_1 v_1 \tilde{k}_2}{\mu_2 \omega} \right) = \left( \frac{\mu_1 v_1}{\mu_2 \omega} \right) \tilde{k}_2$$

Note that these relations for reflection/transmission of  $EM$  waves at normal incidence on a non-conductor/conductor boundary/interface are identical to those obtained for reflection / transmission of  $EM$  waves at normal incidence on a boundary/interface between two non-conductors, except for the replacement of  $\beta$  with a now complex  $\tilde{\beta}$  for the present situation.

For the case of a perfect conductor, the conductivity  $\sigma_c = \infty$  {thus resistivity,  $\rho_c = 1/\sigma_c = 0$ }

$\Rightarrow$  both  $k_2 \approx \kappa_2 \approx \sqrt{\frac{\omega \mu_2 \sigma_c}{2}} = \infty$  and since:  $\tilde{k}_2 = k_2 + i\kappa_2$  then:  $\tilde{k}_2 = \infty + i\infty = \infty(1+i)$

and since:  $\tilde{\beta} \equiv \left( \frac{\mu_1 v_1 \tilde{k}_2}{\mu_2 \omega} \right) = \left( \frac{\mu_1 v_1}{\mu_2 \omega} \right) \tilde{k}_2 \Rightarrow \underline{\underline{\tilde{\beta} = \infty}}$

Thus, for a perfect conductor, we see that:  $\tilde{E}_{o_{refl}} = -\tilde{E}_{o_{inc}}$  and  $\tilde{E}_{o_{trans}} = 0$  and thus for a perfect conductor the reflection and transmission coefficients are:

$$R \equiv \left( \frac{E_{o_{refl}}}{E_{o_{inc}}} \right)^2 = \left| \frac{\tilde{E}_{o_{refl}}}{\tilde{E}_{o_{inc}}} \right|^2 = \left( \frac{\tilde{E}_{o_{refl}}}{\tilde{E}_{o_{inc}}} \right) \left( \frac{\tilde{E}_{o_{refl}}}{\tilde{E}_{o_{inc}}} \right)^* = 1 \quad \text{and:} \quad T = 1 - R = 0$$

We also see that for a perfect conductor, for normal incidence, the reflected wave undergoes a 180° phase shift with respect to the incident wave at the interface/boundary at  $z = 0$  in the  $x$ - $y$  plane.

For the case of a good conductor, the conductivity  $\sigma_c$  is finite-large, but not infinite. The reflection coefficient  $R$  for monochromatic plane  $EM$  waves at normal incidence on a good conductor is not unity, but close to it. {This is why good conductors make good mirrors!}

For a good conductor: 
$$R \equiv \left( \frac{E_{o_{refl}}}{E_{o_{inc}}} \right)^2 = \left| \frac{\tilde{E}_{o_{refl}}}{\tilde{E}_{o_{inc}}} \right|^2 = \left( \frac{\tilde{E}_{o_{refl}}}{\tilde{E}_{o_{inc}}} \right) \left( \frac{\tilde{E}_{o_{refl}}}{\tilde{E}_{o_{inc}}} \right)^* = \left| \frac{1-\tilde{\beta}}{1+\tilde{\beta}} \right|^2 = \left( \frac{1-\tilde{\beta}}{1+\tilde{\beta}} \right) \left( \frac{1-\tilde{\beta}}{1+\tilde{\beta}} \right)^*$$

Where:  $\tilde{\beta} \equiv \left( \frac{\mu_1 v_1 \tilde{k}_2}{\mu_2 \omega} \right) = \left( \frac{\mu_1 v_1}{\mu_2 \omega} \right) \tilde{k}_2$  and:  $\tilde{k}_2 = k_2 + i\kappa_2$ . For a good conductor:  $k_2 \approx \kappa_2 \approx \sqrt{\frac{\omega \mu_2 \sigma_c}{2}}$

Thus: 
$$\tilde{\beta} = \left( \frac{\mu_1 v_1}{\mu_2 \omega} \right) \tilde{k}_2 = \left( \frac{\mu_1 v_1}{\mu_2 \omega} \right) \sqrt{\frac{\omega \mu_2 \sigma_c}{2}} (1+i) = \mu_1 v_1 \sqrt{\frac{\sigma_c}{2\mu_2 \omega}} (1+i)$$

Define:  $\gamma \equiv \mu_1 v_1 \sqrt{\frac{\sigma_c}{2\mu_2 \omega}}$  Then:  $\tilde{\beta} = \gamma(1+i)$

Thus, the reflection coefficient  $R$  for monochromatic plane  $EM$  waves at normal incidence on a good conductor is:

$$R = \left| \frac{\tilde{E}_{o_{refl}}}{\tilde{E}_{o_{inc}}} \right|^2 = \left| \frac{1 - \tilde{\beta}}{1 + \tilde{\beta}} \right|^2 = \left( \frac{1 - \tilde{\beta}}{1 + \tilde{\beta}} \right) \left( \frac{1 - \tilde{\beta}}{1 + \tilde{\beta}} \right)^* = \left( \frac{1 - \gamma - i\gamma}{1 + \gamma + i\gamma} \right) \left( \frac{1 - \gamma + i\gamma}{1 + \gamma - i\gamma} \right) = \left[ \frac{(1 - \gamma)^2 + \gamma^2}{(1 + \gamma)^2 + \gamma^2} \right]$$

with: 
$$\gamma \equiv \mu_1 v_1 \sqrt{\frac{\sigma_c}{2\mu_2 \omega}}$$

Obviously, a small fraction of the normally-incident monochromatic plane  $EM$  wave is transmitted into the good conductor, since  $R < 1$  and  $T = 1 - R$ , i.e.:

$$T = 1 - R = 1 - \left[ \frac{(1 - \gamma)^2 + \gamma^2}{(1 + \gamma)^2 + \gamma^2} \right] \quad (\ll 1)$$

Note that the transmitted wave is exponentially attenuated in the  $z$ -direction; the  $\vec{E}$  and  $\vec{B}$  fields in the good conductor fall to  $1/e$  of their initial  $\{z = 0\}$  values (at/on the interface) after the monochromatic plane  $EM$  wave propagates a distance of one skin depth in  $z$  into the conductor:

$$\delta_{sc}(\omega) \equiv \frac{1}{\kappa_2(\omega)} \approx \sqrt{\frac{2}{\omega \mu_2 \sigma_c}}$$

Note also that the energy associated with the transmitted monochromatic plane  $EM$  wave is ultimately dissipated in the conducting medium as heat.

### Full Maxwell Equations in Matter:

The electromagnetic state of matter at a given observation point  $\vec{r}$  at a given time  $t$  is described by four macroscopic quantities:

- 1.) The volume density of free charge:  $\rho_{free}(\vec{r}, t)$
- 2.) The volume density of electric dipoles:  $\vec{P}(\vec{r}, t) \Leftarrow aka$  electric polarization
- 3.) The volume density of magnetic dipoles:  $\vec{M}(\vec{r}, t) \Leftarrow aka$  magnetization
- 4.) The free electric current/unit area:  $\vec{J}_{free}(\vec{r}, t) \Leftarrow aka$  {free} current density

All four of these quantities are macroscopically averaged - i.e. the microscopic fluctuations due to atomic/molecular makeup of matter have been smoothed out.

The four above quantities are related to the macroscopic  $\vec{E}$  and  $\vec{B}$  fields by the four Maxwell equations for matter (see Physics 435 Lect. Notes 24, p. 14):

1) Gauss' Law:  $\vec{\nabla} \cdot \vec{E} = \frac{\rho_{Tot}}{\epsilon_o} = \frac{1}{\epsilon_o} (\rho_{free} + \rho_{bound})$ , where:  $\rho_{bound} = -\vec{\nabla} \cdot \vec{P}$

Auxiliary relation:  $\vec{D} = \epsilon_o \vec{E} + \vec{P}$  & constitutive relation:  $\vec{D} = \epsilon \vec{E}$

Electric polarization  $\vec{P} = (\epsilon - \epsilon_o) \vec{E} = \epsilon_o \chi_e \vec{E}$ , electric susceptibility  $\chi_e = \left( \frac{\epsilon}{\epsilon_o} - 1 \right)$

$$\vec{\nabla} \cdot \vec{D} = \epsilon_o \vec{\nabla} \cdot \vec{E} + \vec{\nabla} \cdot \vec{P} = \rho_{free}$$

2) No magnetic charges/monopoles:  $\vec{\nabla} \cdot \vec{B} = 0$

Auxiliary relation:  $\vec{H} = \frac{1}{\mu_o} \vec{B} - \vec{M} \Rightarrow \vec{\nabla} \cdot \vec{H} = -\vec{\nabla} \cdot \vec{M}$  & constitutive relation:  $\vec{B} = \mu \vec{H}$

3) Faraday's Law:  $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\mu_o \frac{\partial \vec{H}}{\partial t} - \mu_o \frac{\partial \vec{M}}{\partial t}$

Magnetization:  $\vec{M} - \left( \frac{\mu}{\mu_o} - 1 \right) \vec{H} = \chi_m \vec{H}$ , magnetic susceptibility  $\chi_m = \left( \frac{\mu}{\mu_o} - 1 \right)$

4) Ampere's Law:  $\vec{\nabla} \times \vec{B} = \mu_o \vec{J}_{Tot} + \mu_o \vec{J}_D$  with  $\vec{J}_D = \epsilon_o \frac{\partial \vec{E}}{\partial t}$

Total current density:  $\vec{J}_{Tot} = \vec{J}_{free} + \vec{J}_{bound}^{mag} + \vec{J}_{bound}^P$   $\vec{J}_{bound}^{mag} = \vec{\nabla} \times \vec{M}$   $\vec{J}_{bound}^P = \frac{\partial \vec{P}}{\partial t}$

$$\vec{\nabla} \times \vec{B} = \mu_o \vec{J}_{free} + \mu_o \vec{\nabla} \times \vec{M} + \mu_o \frac{\partial \vec{P}}{\partial t} + \mu_o \epsilon_o \frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla} \times \vec{H} = \mu_o \vec{J}_{free} + \mu_o \frac{\partial \vec{D}}{\partial t}$$

We also have Ohm's Law:  $\vec{J} = \sigma_c \vec{E}$  and the 3 Continuity eqn(s):  $\vec{\nabla} \cdot \vec{J}_\alpha = -\frac{\partial \rho_\alpha}{\partial t}$

For many/most (but not all!!!) physics problems, e.g. in optics/condensed matter physics, materials of interest are frequently non-magnetic (or negligibly magnetic) and have no (free) charge densities present, i.e.  $\rho_{free} = 0$ . If  $\underline{\mu} \simeq \underline{\mu}_o$ , then  $\underline{\vec{M}} = 0$  and thus  $\underline{\vec{H}} = \underline{\vec{B}}/\underline{\mu}_o$  in such non-magnetic materials.

Then Maxwell's equations in matter, for  $\rho_{free} = 0$  and  $\underline{\vec{M}} = 0$  reduce to:

1) Gauss' Law:  $\underline{\vec{\nabla}} \cdot \underline{\vec{D}} = 0$  or  $\underline{\vec{\nabla}} \cdot \underline{\vec{E}} = -\frac{1}{\epsilon_o} \underline{\vec{\nabla}} \cdot \underline{\vec{P}} = \rho_{free} / \epsilon_o$

2) No magnetic charges:  $\underline{\vec{\nabla}} \cdot \underline{\vec{B}} = 0$

3) Faraday's Law:  $\underline{\vec{\nabla}} \times \underline{\vec{E}} = -\frac{\partial \underline{\vec{B}}}{\partial t}$

4) Ampere's Law:  $\underline{\vec{\nabla}} \times \underline{\vec{B}} = \mu_o \epsilon_o \frac{\partial \underline{\vec{E}}}{\partial t} + \mu_o \frac{\partial \underline{\vec{P}}}{\partial t} + \mu_o \underline{\vec{J}}_{free}$

We also have Ohm's Law  $\underline{\vec{J}}_{free} = \sigma_c \underline{\vec{E}}$  and the Continuity eqn.  $\underline{\vec{\nabla}} \cdot \underline{\vec{J}}_{free} = 0$  {here}.

Then applying the curl operator to Faraday's Law:

$$\underline{\vec{\nabla}} \times (\underline{\vec{\nabla}} \times \underline{\vec{E}}) = -\frac{\partial}{\partial t} (\underline{\vec{\nabla}} \times \underline{\vec{B}}) = -\mu_o \epsilon_o \frac{\partial^2 \underline{\vec{E}}}{\partial t^2} - \mu_o \frac{\partial^2 \underline{\vec{P}}}{\partial t^2} - \mu_o \frac{\partial \underline{\vec{J}}_{free}}{\partial t} = \underline{\vec{\nabla}} (\underline{\vec{\nabla}} \cdot \underline{\vec{E}}) - \nabla^2 \underline{\vec{E}} = \frac{1}{\epsilon_o} \underline{\vec{\nabla}} \rho_{bound} - \nabla^2 \underline{\vec{E}}$$

We thus obtain the inhomogeneous wave equation:

$$\nabla^2 \underline{\vec{E}} - \frac{1}{c^2} \frac{\partial^2 \underline{\vec{E}}}{\partial t^2} = \underbrace{\frac{1}{\epsilon_o} \nabla \rho_{bound} + \mu_o \frac{\partial^2 \underline{\vec{P}}}{\partial t^2} + \mu_o \frac{\partial \underline{\vec{J}}_{free}}{\partial t}}_{\text{source terms}} \quad \{\text{and a similar/analogous one for } \underline{\vec{B}} \}$$

For nonconducting/poorly-conducting media, i.e. insulators/dielectrics, the first two terms on the RHS of the above equation are important – e.g. they explain many optical effects such as dispersion (wavelength/frequency-dependence of the index of refraction), absorption, double – refraction/bi-refringence, optical activity, . . . .

Note that the  $\underline{\vec{\nabla}} \rho_{bound} = -\underline{\vec{\nabla}} (\underline{\vec{\nabla}} \cdot \underline{\vec{P}})$  term is often zero, e.g. if the electric polarization  $\underline{\vec{P}}$  is uniform,

or since:  $\underline{\vec{\nabla}} \cdot \underline{\vec{P}} = \frac{\partial P_x}{\partial x} + \frac{\partial P_y}{\partial y} + \frac{\partial P_z}{\partial z}$  and  $\underline{\vec{\nabla}} = \frac{\partial}{\partial x} \hat{x} + \frac{\partial}{\partial y} \hat{y} + \frac{\partial}{\partial z} \hat{z}$

e.g. for  $\underline{\vec{P}} \propto \underline{\vec{E}}$  (i.e.  $\underline{\vec{P}}$  proportional to  $\underline{\vec{E}}$ ) where:  $\underline{\vec{E}}(z, t) = E_o \cos(kz - \omega t + \delta) \hat{x}$

For good conductors (e.g. metals), the conduction term  $\mu_0 \frac{\partial \vec{J}_{free}}{\partial t} = \mu_0 \sigma_c \frac{\partial \vec{E}}{\partial t}$  is the most important, because it explains the opacity of metals (e.g. in the visible light region) and also explains the high reflectance of metals.

All source terms on the RHS of the above inhomogeneous wave equation are of importance for semiconductors – however a proper/more complete physics description of *EM* wave propagation in semiconductors also requires the addition of quantum theory for rigorous treatment...