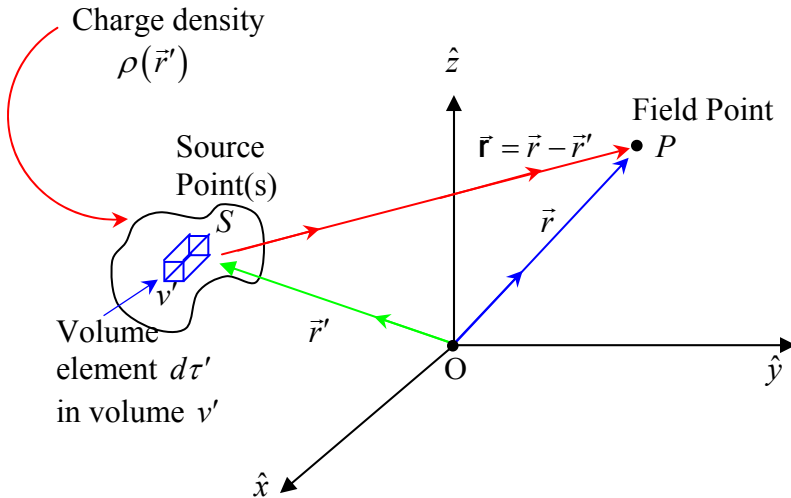


LECTURE NOTES 7

LAPLACE'S EQUATION

As we have seen in previous lectures, very often the primary task in an electrostatics problem is e.g. to determine the electric field $\vec{E}(\vec{r})$ of a given stationary/static charge distribution – e.g. via Coulomb's Law:



$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{v'} \frac{\hat{r}}{r^2} \rho(\vec{r}') d\tau' \quad \vec{r} = \vec{r} - \vec{r}' \quad \hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|}$$

$$|\vec{r}| = |\vec{r} - \vec{r}'| = \sqrt{(x_p - x_s)^2 + (y_p - y_s)^2 + (z_p - z_s)^2}$$

Oftentimes $\rho(\vec{r}')$ is complicated, and *analytic* calculation of $\vec{E}(\vec{r})$ is painful / tedious (or just plain hard). (Numerical integration on a computer is likely faster/easier. . .)

Oftentimes it is easier to *first* calculate the potential $V(\vec{r})$, and *then* use $\vec{E}(\vec{r}) = -\vec{\nabla}V(\vec{r})$

$$\text{Here } V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{v'} \frac{1}{r} \rho(\vec{r}') d\tau'$$

But even doing this integral analytically often can be very challenging. . .

Furthermore, often in problems involving conductors, $\rho(\vec{r}')$ may not *a priori* (i.e. beforehand) be known! Charge is free to move around, and often only the total *free* charge Q_{free} is controlled / known in the problem.

In such cases, it is usually better to recast the problem in DIFFERENTIAL form, using Poisson's equation:

$$\vec{\nabla} \cdot \vec{E}(\vec{r}) = -\vec{\nabla} \cdot \vec{\nabla}V(\vec{r}) = -\nabla^2 V(\vec{r}) = \frac{\rho(\vec{r})}{\epsilon_0}$$

Or:
$$\boxed{\nabla^2 V(\vec{r}) = -\frac{\rho(\vec{r})}{\epsilon_0}} \leftarrow \text{Poisson's Equation}$$

Poisson's equation, together with the *boundary conditions* associated with the value(s) allowed for $V(\vec{r})$ e.g. on various conducting surfaces, or at $r = \infty$, etc. enables one to uniquely determine $V(\vec{r})$ (we'll see how / why shortly. . .).

Now the Poisson Equation is an inhomogeneous second-order differential equation – its solution consists of a particular solution for the inhomogeneous term (RHS of Poisson's Equation) plus the general solution for the homogeneous second-order differential equation:

$$\boxed{\nabla^2 V(\vec{r}) = 0} \leftarrow \text{Laplace's Equation}$$

commensurate with the boundary conditions for the specific problem at hand.

Very often, in fact, we are interested in finding the potential $V(\vec{r})$ in a charge-free region, containing no electric charge, i.e. where $\rho(\vec{r}') = 0$.

If $\rho(\vec{r}') = 0$, then $\nabla^2 V(\vec{r}) = 0$ and the TRIVIAL solution is $V(\vec{r}) = 0 \forall \vec{r}$, which is boring / useless!

We seek physically meaningful / non-trivial solutions $V(\vec{r}) \neq 0$ that satisfy $\nabla^2 V(\vec{r}) = 0$ and the boundary conditions on $V(\vec{r})$ for a given physical problem.

Now, before we go any further on this discussion, let's back up a bit and take a (very) broad generalized MATHEMATICAL view (or approach) to find $V(\vec{r})$.

First, let's simplify the discussion, by talking about one-dimensional problems:

If $\rho(x) = 0$, Laplace's Equation in one-dimension becomes (in rectangular/Cartesian coordinates):

$$\nabla^2 V(\vec{r}) = 0 \Rightarrow \boxed{\frac{d^2 V(x)}{dx^2} = 0} \leftarrow \text{Note the } \underline{\text{total}} \text{ (not partial) derivative with regards to } x.$$

Integrating this equation (both sides) once, we have:

$$\int \frac{d^2 V(x)}{dx^2} dx = \int \frac{d}{dx} \left(\frac{dV(x)}{dx} \right) dx = \int d \left(\frac{dV}{dx} \right) = \frac{dV(x)}{dx} = \int dx = m = 1^{\text{st}} \text{ constant of integration}$$

Then: $\int \frac{dV(x)}{dx} dx = \int m dx = m \int dx$

Or: $\int dV(x) = V(x) = mx + b \leftarrow 2^{\text{nd}} \text{ constant of integration}$

So: $V(x) = b + mx$ (equation for a straight line) is the general solution for $\frac{d^2V(x)}{dx^2} = 0$.
 y-intercept \rightarrow b slope \rightarrow m

Depending on the boundary conditions for the problem e.g. suppose $V(x=5) = 0$ Volts and $V(x=1) = 4$ Volts, then these two boundary conditions (together) uniquely specify what b and m must be:

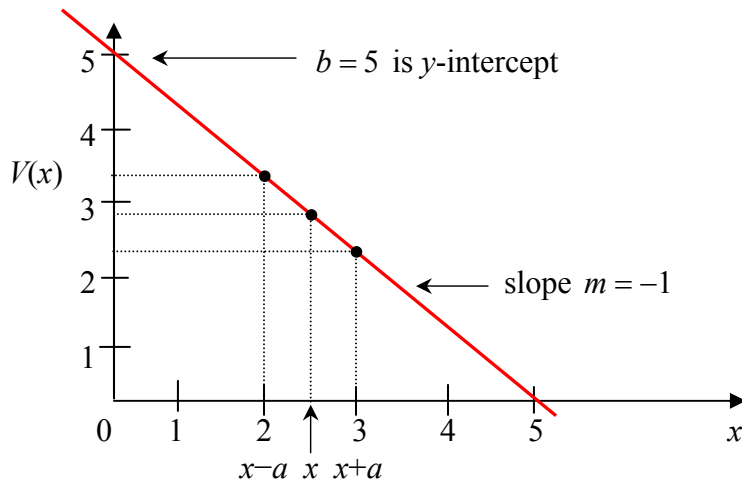
$V(x) = b + mx$ \leftarrow equation for a straight line
 y-intercept \rightarrow b slope \rightarrow m

$$V(x=5) = 0 = b + 5m \quad \rightarrow \quad b = -5m$$

$$V(x=1) = 4 = b + 1m \quad \rightarrow \quad 4 = -5m + 1m = -4m$$

$$\therefore V(x) = 5 - 1x \quad \text{or:} \quad m = -1 \quad \text{and} \quad b = 5.$$

$V(x) = 5 - 1x$ is the equation of a straight line for this problem.



General features of 1-D Laplace's Equation $\nabla^2 V(x) = 0$ and potential $V(x)$:

1. From above one-dimensional case $V(x) = b + mx$ (general solution = straight line eqn.) we can see that:

$$V(x) \text{ is the } \underline{\text{average}} \text{ of } V(x+a) \text{ and } V(x-a) \text{ i.e. } V(x) = \frac{1}{2}\{V(x+a) + V(x-a)\}$$

\Rightarrow Laplace's Equation is a kind of averaging instruction

- solutions of $V(x)$ are as "boring" as possible, but fit endpoints (boundary conditions) properly.

This may be "obvious" in one-dimension, but is also true / also holds in 2-D and 3-D cases of $\nabla^2 V(\vec{r}) = 0$.

2. $\nabla^2 V(\vec{r})$ tolerates / allows NO local maxima or minima – extrema must occur at endpoints
 i.e. $\nabla^2 V(\vec{r})$ requires second derivative(s) of $V(\vec{r})$ to be zero.
 - Not a proof, because e.g. \exists *fcns*(x) where the second derivative vanishes other than at endpoints
 - e.g. $f(x) = x^4$ (has a minimum at $x = 0$).

Laplace's Equation in Two Dimensions (in Rectangular/Cartesian Coordinates)

If $V = V(x, y)$ then $\nabla^2 V = 0 \Rightarrow \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$

n.b. now have partial derivatives of $V(\vec{r})$.

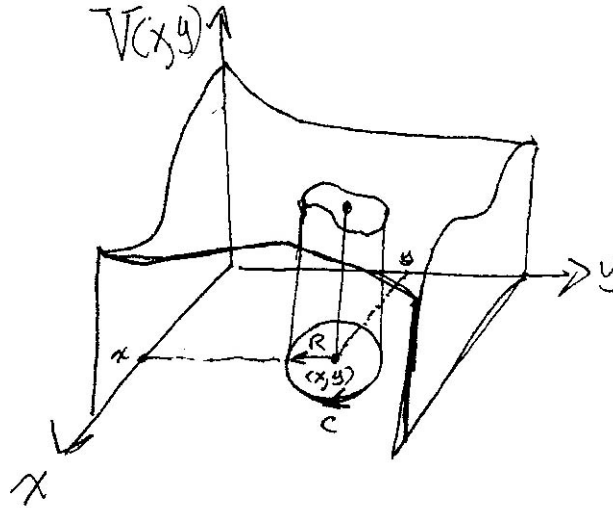
Because $\nabla^2 V = 0 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) V(x, y)$ now contains partial derivatives, the general solution

does not contain just two arbitrary constants or any finite number - \exists an infinite number of possible solutions (in general)

– most general solution is a linear combination of harmonic functions (sine and cosine functions of x and y in rectangular coordinates and other functions (Bessel Functions) in cylindrical coordinates).

Nevertheless, $V(x, y)$ will still wind up being the average value of V around a point (x, y) within a circle of radius R centered on the point (x, y) .

The Method of Relaxation - Iterative Computer Algorithm for Finding $V(x, y)$:



$$V(x, y) = \frac{1}{2\pi R} \oint_{\text{circle } C \text{ of radius } R \text{ centered on } (x, y)} V(\vec{r}) dl$$

- Start with $V(x, y)$ as specified on boundary (fixed)
- Choose reasonable “interpolated” values of $V(x, y)$ (from boundary) on interior (x, y) points away from the boundaries.
- 1st pass reassigns $V(x, y) =$ average value at interior point (x, y) of its nearest neighbors.
- 2nd pass repeats this process . . .
- 3rd pass repeats this process . . .
- etc.

After few iterations, $V(x, y)$ of n^{th} iteration settles down, e.g. when:

$$\Delta V(x, y) = \left| \overbrace{V_n(x, y)}^{\text{iteration } n} - \overbrace{V_{n-1}(x, y)}^{\text{iteration } n-1} \right| \leq \text{tolerance}$$

then QUIT iterating, $V(x, y)$ is determined after n^{th} iteration is “good enough”.

$V(x, y)$ again will have no local maxima or minima – all extrema will occur on boundaries.

$\nabla^2 V(x, y) = 0$ has solution $V(x, y)$ which is the most featureless function – as smooth as possible.

Laplace's Equation in Three Dimensions

Can't draw this on 2-D sheet of paper (because now this is a 4 dimensional problem!), but:

$V(x, y, z) = V(\vec{r}) =$ average value of V over a spherical surface of radius R centered on \vec{r} .

i.e.
$$V(\vec{r}) = \frac{1}{4\pi R^2} \oint_{\text{sphere at } \vec{r} \text{ of radius } R} V da$$

Again $V(\vec{r})$ will have no local maxima or minima

- all extrema must occur at boundaries of problem (see work-through proof in Griffiths, p. 114)
- The average potential produced by a collection of charges, averaged over a sphere of radius R is equal to the value of the potential at the center of that sphere!

Boundary Conditions on the Potential $V(\vec{r})$

Dirichlet Boundary Conditions on $V(\vec{r})$:

$V(\vec{r})$ itself is specified (somewhere) on the boundary - i.e. the value of $V(\vec{r})$ is specified (somewhere) on the boundary.

Neumann Boundary Conditions on $V(\vec{r})$:

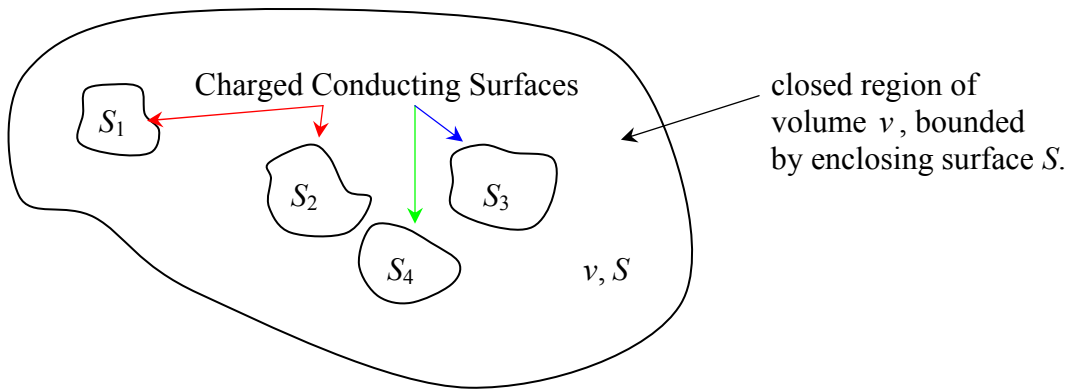
The normal derivative of $V(\vec{r})$ is specified somewhere on the boundary - i.e.

$\vec{\nabla}V(\vec{r}) \cdot \hat{n} = -E^\perp(\vec{r})$ is specified somewhere on the boundary.

Uniqueness Theorem(s):

Suppose we have two solutions of Laplace’s Equation, $V_1(\vec{r})$ and $V_2(\vec{r})$, each satisfying the same boundary condition(s). They can at most differ by a constant. (n.b. Only relative differences in potential are important / physically meaningful!)

Proof: Consider a closed region of space with volume v which is exterior to charged conducting surfaces $S_1, S_2, S_3, \dots S_n$ that are responsible for generating the potential V . The volume v is bounded (outside) by the surface S .



Suppose we have two solutions $V_1(r)$ and $V_2(r)$ both satisfying $\nabla^2 V(\vec{r}) = 0$ i.e. $\nabla^2 V_1(\vec{r}) = 0$ and $\nabla^2 V_2(\vec{r}) = 0$ in the charge-free region(s) of the volume v .

$V_1(r)$ and $V_2(r)$ satisfy either Dirichlet boundary conditions or satisfy Neumann boundary conditions $\vec{\nabla} V(\vec{r}) \cdot \hat{n}$ on the surfaces $S_1, S_2, S_3, \dots S_n$. We also demand that $V(r)$ be finite at $r = \infty$.

Let us define: $V_\Delta(\vec{r}) \equiv V_1(\vec{r}) - V_2(\vec{r}) =$ difference in the two potential solutions at the point \vec{r} .

Now both $\nabla^2 V_1(\vec{r}) = 0$ and $\nabla^2 V_2(\vec{r}) = 0$.

Thus: $\nabla^2 V_\Delta(\vec{r}) = \nabla^2 (V_1(\vec{r}) - V_2(\vec{r})) = \underbrace{\nabla^2 V_1(\vec{r})}_{=0} - \underbrace{\nabla^2 V_2(\vec{r})}_{=0} = 0$ Note that: $\nabla^2 V(\vec{r}) = \vec{\nabla} \cdot (\vec{\nabla} V(\vec{r}))$

The potentials $V_{i=1,2}$ are uniquely specified on charged (equipotential) surfaces $S_1, S_2, S_3, \dots S_n$ in volume v .

Now apply the divergence theorem to the quantity $(V_\Delta \vec{\nabla} V_\Delta)$:

$$\int_v \vec{\nabla} \cdot (V_\Delta \vec{\nabla} V_\Delta) d\tau = \int_{S_1+S_2+S_3+\dots+S_n}^{S+} (V_\Delta \vec{\nabla} V_\Delta) \cdot d\vec{A} = \int_{S_1+S_2+S_3+\dots+S_n}^{S+} V_\Delta (-\vec{E}_\Delta) \cdot d\vec{A}$$

↗ {define $\vec{E}_\Delta(\vec{r}) \equiv -\vec{\nabla} V_\Delta(\vec{r})$ }

integrate over enclosing volume v surface integral over ALL surfaces in v

Then:

$$- \int_{S_1+S_2+S_3+\dots+S_n}^{S+} V_{\Delta} (\vec{E}_{\Delta} \cdot d\vec{A}) = - \int_S V_{\Delta} (\vec{E}_{\Delta} \cdot d\vec{A}_S) - \int_{S_1} V_{\Delta} (\vec{E}_{\Delta} \cdot d\vec{A}_{S_1}) - \int_{S_2} V_{\Delta} (\vec{E}_{\Delta} \cdot d\vec{A}_{S_2}) - \dots - \int_{S_n} V_{\Delta} (\vec{E}_{\Delta} \cdot d\vec{A}_{S_n})$$

Recognizing that:

1. The conducting surfaces $S_1, S_2, S_3, \dots, S_n$, are equipotentials.
Thus $V_{\Delta}(\vec{r}) = V_1(\vec{r}) - V_2(\vec{r})$ (= a constant on surfaces $S_1, S_2, S_3, \dots, S_n$) must = 0 at/on those surfaces!!!
2. The volume v is arbitrary, so let's choose volume $v \rightarrow \infty$, and thus surface area $S \rightarrow \infty$ as well.
3. $\int_{S_i} \vec{E}_{\Delta} \cdot d\vec{A}_i = \Phi_{E_i}$ = electric flux through i^{th} surface.
4. $V_{\Delta}(r \rightarrow \infty) = V_1(r \rightarrow \infty) - V_2(r \rightarrow \infty)$ (= constant on surface $S \rightarrow \infty$) must = 0
because $V_1(r \rightarrow \infty) = V_2(r \rightarrow \infty)$.

$$\therefore \int_{\substack{v \\ \text{all space}}} \vec{\nabla} \cdot (V_{\Delta} \vec{\nabla} V_{\Delta}) d\tau = - \underbrace{V_{\Delta}^S}_{=0} \int_{\substack{S \\ \text{all space}}} \vec{E}_{\Delta} \cdot d\vec{A}_S - \underbrace{V_{\Delta}^{S_1}}_{=0} \int_{S_1} \vec{E}_{\Delta} \cdot d\vec{A}_{S_1} - \underbrace{V_{\Delta}^{S_2}}_{=0} \int_{S_2} \vec{E}_{\Delta} \cdot d\vec{A}_{S_2} - \dots - \underbrace{V_{\Delta}^{S_n}}_{=0} \int_{S_n} \vec{E}_{\Delta} \cdot d\vec{A}_{S_n}$$

$\underbrace{\hspace{10em}}_{=\Phi_E^S} \quad \underbrace{\hspace{10em}}_{=\Phi_E^{S_1}} \quad \underbrace{\hspace{10em}}_{=\Phi_E^{S_2}} \quad \underbrace{\hspace{10em}}_{=\Phi_E^{S_n}}$

$$\text{Thus: } \int_{\substack{v \\ \text{all space}}} \vec{\nabla} \cdot (V_{\Delta} \vec{\nabla} V_{\Delta}) d\tau = 0$$

$$\text{However, using the identity } \vec{\nabla} \cdot (V_{\Delta} \vec{\nabla} V_{\Delta}) = V_{\Delta} (\nabla^2 V_{\Delta}) + \underbrace{(\vec{\nabla} V_{\Delta}) \cdot \vec{\nabla} V_{\Delta}}_{=0}$$

$$\begin{aligned} \text{Then: } \int_{\substack{v \\ \text{all space}}} \vec{\nabla} \cdot (V_{\Delta} \vec{\nabla} V_{\Delta}) d\tau &= \int_{\substack{v \\ \text{all space}}} V_{\Delta} \underbrace{(\nabla^2 V_{\Delta})}_{=0} d\tau + \int_{\substack{v \\ \text{all space}}} (\vec{\nabla} V_{\Delta})^2 d\tau = 0 \\ &= \int_{\substack{v \\ \text{all space mathematically} \\ \geq 0}} \underbrace{(\vec{\nabla} V_{\Delta})^2}_{\geq 0} d\tau = \int_{\substack{v \\ \text{all space}}} (\vec{\nabla} V_{\Delta} \cdot \vec{\nabla} V_{\Delta}) d\tau = 0 \end{aligned}$$

The only way $\int_V \underbrace{(\vec{\nabla} V_{\Delta})^2}_{\substack{\text{mathematically} \\ \geq 0}}$ $d\tau = 0$ is iff (i.e. if and only if) the integrand

$$(\vec{\nabla} V_{\Delta}(\vec{r}))^2 = (\vec{\nabla} V_{\Delta}(\vec{r}) \cdot \vec{\nabla} V_{\Delta}(\vec{r})) = 0.$$

$$\text{If } (\vec{\nabla} V_{\Delta}(\vec{r}))^2 = (\vec{\nabla} V_{\Delta}(\vec{r}) \cdot \vec{\nabla} V_{\Delta}(\vec{r})) = 0,$$

Then: $\vec{\nabla} V_{\Delta}(\vec{r})$ itself must be = 0 (i.e. $\vec{A}(\vec{r}) \cdot \vec{A}(\vec{r}) = 0 \Rightarrow \vec{A}(\vec{r}) \equiv 0$) for all points (\vec{r}) in volume v .

If $\vec{\nabla} V_{\Delta}(\vec{r}) = 0$ for all points \vec{r} in volume v , then $V_{\Delta}(\vec{r})$ = (same) constant at all points in volume v . $\therefore V_{\Delta}(\vec{r}) = V_1(\vec{r}) - V_2(\vec{r}) = \text{constant at all points in volume } v$.

Dirichlet Boundary Conditions (V specified on $S_1, S_2, S_3, \dots, S_n$)

If $V_1(r)$ and $V_2(r)$ are specified on the surfaces $S_1, S_2, S_3, \dots, S_n$ in the volume v enclosed by surface S (Dirichlet boundary conditions), then: $V_\Delta(\vec{r}) = V_1(\vec{r}) - V_2(\vec{r}) = 0$ (i.e. problem is over-determined).

← *must = 0!*

$\therefore V_\Delta(\vec{r}) = 0$ throughout the volume v and $V_1(\vec{r}) = V_2(\vec{r})$ throughout the volume v .

i.e. the two solutions $V_1(r)$ and $V_2(r)$ for $\nabla^2 V(\vec{r}) = 0$ are identical – there is only one unique solution.

Neumann Boundary Conditions (E^\perp specified on $S_1, S_2, S_3, \dots, S_n$)

If $\bar{\nabla}V_1 \cdot \hat{n} = -\vec{E}_1^\perp$ and $\bar{\nabla}V_2 \cdot \hat{n} = -\vec{E}_2^\perp$ are specified on the surfaces $S_1, S_2, S_3, \dots, S_n$ in the volume v enclosed by surface S (Neumann boundary conditions), then $\bar{\nabla}V_\Delta(\vec{r}) = \bar{\nabla}V_1(\vec{r}) - \bar{\nabla}V_2(\vec{r}) = 0$ at all points in volume v and $\bar{\nabla}V_\Delta \cdot \hat{n} = 0$.

Then $V_\Delta(\vec{r}) = V_1(\vec{r}) - V_2(\vec{r}) = \text{constant}$, but is not necessarily = 0 !!!

Here, solutions $V_1(r)$ and $V_2(r)$ can differ, but only by a constant V_o .

e.g. $V_1(\vec{r}) = V_2(\vec{r}) + V_o \Rightarrow$ problem is NOT over-determined for $V(\vec{r})$.

($\vec{E}(\vec{r})$ is over-determined / unique, but not $V(\vec{r})$).

Physical Example:

<p>The Parallel Plate Capacitor: $E = \Delta V/d = 100 \text{ V/m}$</p>	$\frac{+100 \text{ V}}{\downarrow \vec{E}}$ <hr style="width: 50%; margin: 0 auto;"/> $\frac{0 \text{ V}}{\uparrow}$	$\frac{\downarrow}{d = 1 \text{ m}}$ <hr style="width: 50%; margin: 0 auto;"/> $\frac{\uparrow}{}$
<p>Or:</p>	$E = \Delta V/d = 100 \text{ V/m}$	$\frac{+500 \text{ V}}{\downarrow \vec{E}}$ <hr style="width: 50%; margin: 0 auto;"/> $\frac{+400 \text{ V}}{\uparrow}$
<p>$\Delta V = 100 \text{ V}$ in both cases – thus E-field is same/identical in both cases!</p>		

Solving Laplace's Equation ($\nabla^2 V(\vec{r}) = 0$) in 3-D, 2-D and 1-D Situations

In general, when solving the potential $V(\vec{r})$ problems in 3 (or less) dimensions, first note the symmetries associated with the problem. Then, if you have:

$$\left\{ \begin{array}{l} \text{Rectangular} \\ \text{Cylindrical} \\ \text{Spherical} \end{array} \right\} \text{ Symmetry} \Rightarrow \text{Solve Problem Using} \left\{ \begin{array}{l} \text{Rectangular} \\ \text{Cylindrical} \\ \text{Spherical} \end{array} \right\} \text{ Coordinates}$$

In 2-D and 3-D problems, the general solutions to $\nabla^2 V(\vec{r}) = 0$ are the harmonic functions (an ∞ -series solution, in principle) e.g. of sines and cosines, Bessel functions, or Legendre Polynomials and/or Spherical Harmonics.

The boundary conditions / symmetries will select a subset of the ∞ -solutions.

We will now work through derivations of finding solutions to Laplace's Equations in 3-dimensions in rectangular (i.e. Cartesian) coordinates, cylindrical coordinates, and spherical coordinates. We will also use / show the method of separation of variables.

Laplace's Equation $\nabla^2 V(x, y, z) = 0$ and Potential Problems with Rectangular Symmetry (Rectangular / Cartesian coordinates)

In Three Dimensions: Solve Laplace's equation in rectangular / Cartesian coordinates:

$$\nabla^2 V(x, y, z) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) V(x, y, z) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

The solutions of $\nabla^2 V = 0$ in rectangular coordinates are known as harmonic functions (i.e. sines and cosines) (\rightarrow Fourier Series Solutions).

It is usually (but not always) possible to find a solution to the Laplace Equation, $\nabla^2 V = 0$ which also satisfies the boundary conditions, via separation of variables technique, i.e. try a product solution of the form:

$$V(x, y, z) = X(x)Y(y)Z(z)$$

where: $\left\{ \begin{array}{l} X(x) \\ Y(y) \\ Z(z) \end{array} \right\}$ are functions only of $\left\{ \begin{array}{l} x \\ y \\ z \end{array} \right\}$ respectively.

Then: $\nabla^2 V(x, y, z) = 0 \Rightarrow \frac{\partial^2 V(x, y, z)}{\partial x^2} + \frac{\partial^2 V(x, y, z)}{\partial y^2} + \frac{\partial^2 V(x, y, z)}{\partial z^2} = 0$

But: $V(x, y, z) = X(x)Y(y)Z(z)$

Thus:
$$\frac{\partial^2 X(x)Y(y)Z(z)}{\partial x^2} + \frac{\partial^2 X(x)Y(y)Z(z)}{\partial y^2} + \frac{\partial^2 X(x)Y(y)Z(z)}{\partial z^2} = 0$$

$$= Y(y)Z(z) \frac{\partial^2 X(x)}{\partial x^2} + X(x)Z(z) \frac{\partial^2 Y(y)}{\partial y^2} + X(x)Y(y) \frac{\partial^2 Z(z)}{\partial z^2} = 0$$

Now divide both sides of the above equation by $X(x)Y(y)Z(z)$:

Then:
$$= \frac{1}{X(x)} \frac{\partial^2 X(x)}{\partial x^2} + \frac{1}{Y(y)} \frac{\partial^2 Y(y)}{\partial y^2} + \frac{1}{Z(z)} \frac{\partial^2 Z(z)}{\partial z^2} = 0$$

But:
$$= \underbrace{\frac{1}{X(x)} \frac{\partial^2 X(x)}{\partial x^2}}_{\substack{\text{fcn(x) only} \\ \text{independent of } y, z \\ = C_1}} + \underbrace{\frac{1}{Y(y)} \frac{\partial^2 Y(y)}{\partial y^2}}_{\substack{\text{fcn(y) only} \\ \text{independent of } x, z \\ = C_2}} + \underbrace{\frac{1}{Z(z)} \frac{\partial^2 Z(z)}{\partial z^2}}_{\substack{\text{fcn(z) only} \\ \text{independent of } x, y \\ = C_3}} = 0 \quad \text{i.e. } C_1 + C_2 + C_3 = 0$$

True for all points (x, y, z) in volume v of problem.

The only way the above equation can be true for all points (x, y, z) in volume v is if:

$\frac{1}{X(x)} \frac{\partial^2 X(x)}{\partial x^2} = \text{constant } C_1 \Rightarrow$	$\frac{d^2 X(x)}{dx^2} - C_1 X(x) = 0$	#1
$\frac{1}{Y(y)} \frac{\partial^2 Y(y)}{\partial y^2} = \text{constant } C_2 \Rightarrow$	$\frac{d^2 Y(y)}{dy^2} - C_2 Y(y) = 0$	#2
$\frac{1}{Z(z)} \frac{\partial^2 Z(z)}{\partial z^2} = \text{constant } C_3 \Rightarrow$	$\frac{d^2 Z(z)}{dz^2} - C_3 Z(z) = 0$	#3

Note total derivatives now!!!

Subject to the constraint: $C_1 + C_2 + C_3 = 0$

Can now solve 3 ORDINARY 1-D differential equations, #1-3, which are subject to $C_1 + C_2 + C_3 = 0$,
 PLUS the specific Dirichlet / Neumann boundary conditions for the problem on either $V(x, y, z)$ or $\bar{\nabla} V(x, y, z) \cdot \hat{n}$ at surfaces for this 3-D problem.

Essentially, we have replaced the 3-D problem with three 1-D problems, and the constraint $C_1 + C_2 + C_3 = 0$.

* If one has a 2-D rectangular coordinate problem ($\nabla^2 V(x, y) = 0$), then $V(x, y) = X(x)Y(y)$ (only).

$$\left\{ \begin{array}{l} \frac{1}{X(x)} \frac{d^2 X(x)}{dx^2} = C_1 \Rightarrow \frac{d^2 X(x)}{dx^2} - C_1 X(x) = 0 \\ \frac{1}{Y(y)} \frac{d^2 Y(y)}{dy^2} = C_2 \Rightarrow \frac{d^2 Y(y)}{dy^2} - C_2 Y(y) = 0 \end{array} \right.$$

Subject to the constraint: $C_1 + C_2 = 0$, i.e. $C_1 = -C_2$.

Plus BC's: either on $V(x, y)$ or $\bar{\nabla} V(x, y) \cdot \hat{n}$ for 2-D problem.

* If one has a 1-D rectangular coordinate problem ($\nabla^2 V(x) = 0$), then $V(x) = X(x)$ (only).

$$\frac{d^2 V(x)}{dx^2} = 0 \Rightarrow \frac{d^2 X(x)}{dx^2} = 0 \Rightarrow \frac{1}{X(x)} \frac{d^2 X(x)}{dx^2} = 0 = C_1$$

$$\frac{d^2 X(x)}{dx^2} = 0 \Rightarrow X(x) = V(x) = ax + b \text{ is } \underline{\text{the}} \text{ 1-D general solution.}$$

For 1-D problem ($\nabla^2 V(x) = 0$), only need to solve one ordinary differential equation subject to the constraint

$$C_1 = 0 \text{ and BC's on either } V(x) \text{ or } \frac{dV(x)}{dx}.$$

The General Solution $V(x, y, z) = X(x)Y(y)Z(z)$ **for** $\nabla^2 V(x, y, z) = 0$
in Rectangular Coordinates

Since we have the constraint $C_1 + C_2 + C_3 = 0$, at least one of the C_i 's ($i = 1, 2$ or 3) must be less than zero.

Let us "choose" $C_1 = -\alpha^2$, $C_2 = -\beta^2$, $C_3 = \gamma^2$

Then: $C_1 + C_2 + C_3 = 0$
 $-\alpha^2 - \beta^2 + \gamma^2 = 0$ or: $\alpha^2 + \beta^2 = \gamma^2$

The boundary conditions on the surfaces will define α and β , and hence define γ .

IMPORTANT NOTE:

The geometry ($x - y - z$) of the problem and the boundary conditions dictate whether:

$$C_1 > 0 \text{ or } C_1 < 0$$

$$C_2 > 0 \text{ or } C_2 < 0$$

$$C_3 > 0 \text{ or } C_3 < 0$$

i.e. have sine / cosine type solutions vs. sinh / cosh (or e^x , e^{-x}) type solutions for x, y, z .

Then the General Solution is (for above choice of $C_1 = -\alpha^2$, $C_2 = -\beta^2$, $C_3 = \gamma^2$):

$$V(x, y, z) = \sum_{m,n=0}^{\infty} A_{mn} \underbrace{\sin(\alpha_n x)}_{\text{could be cos}} \underbrace{\sin(\beta_m y)}_{\text{could be cos}} \underbrace{\sinh(\gamma_{mn} z)}_{\substack{\text{could be cosh} \\ = \sqrt{\alpha_n^2 + \beta_m^2}}}$$

solutions:

$$\begin{aligned} &+ \sum_{m,n=0}^{\infty} B_{mn} \cos(\alpha_n x) \cos(\beta_m y) \underbrace{\sinh(\gamma_{mn} z)}_{= \sqrt{\alpha_n^2 + \beta_m^2}} \\ &+ \sum_{m,n=0}^{\infty} C_{mn} \sin(\alpha_n x) \sin(\beta_m y) \underbrace{\cosh(\gamma_{mn} z)}_{= \sqrt{\alpha_n^2 + \beta_m^2}} \\ &+ \sum_{m,n=0}^{\infty} D_{mn} \cos(\alpha_n x) \cos(\beta_m y) \underbrace{\cosh(\gamma_{mn} z)}_{= \sqrt{\alpha_n^2 + \beta_m^2}} \end{aligned}$$

$$\text{n.b.} \quad \cosh(x) = \frac{1}{2}(e^x + e^{-x}) \quad \sinh(x) = \frac{1}{2}(e^x - e^{-x})$$

$$\text{n.b.} \quad \sin(x) = \frac{1}{2i}(e^{ix} - e^{-ix}) \quad \cos(x) = \frac{1}{2}(e^{ix} + e^{-ix}) \quad i \equiv \sqrt{-1}$$

$$\text{n.b.} \quad e^{ix} = \cos(x) + i \sin(x) \quad e^{-ix} = \cos(x) - i \sin(x)$$

$$\text{n.b.} \quad e^x = \cosh(x) + \sinh(x) \quad e^{-x} = \cosh(x) - \sinh(x)$$

The BC's and symmetries will determine which of the coefficients A_{mn} , B_{mn} , C_{mn} , $D_{mn} = 0$.

We solve for the non-zero coefficients A_{pq} , B_{pq} , C_{pq} and D_{pq} by taking inner products.

i.e. we multiply $V(x, y, z) = \sum(\text{stuff})$ by e.g. $\sin(\alpha_p x) \sin(\beta_q y)$ to project out the p - q th component

(i.e. we use the orthogonality properties of the individual terms in $\sin(\)$ and $\cos(\)$ Fourier Series.)

And then integrate over the relevant intervals in x and y :

e.g.

$$\begin{aligned} & \int_0^{x_0} \int_0^{y_0} V(x, y) \sin(\alpha_p x) \sin(\beta_q y) dx dy \\ &= \int_0^{x_0} \int_0^{y_0} \left\{ \left(\sum_{m,n=0}^{\infty} A_{mn} \sin(\alpha_n x) \sin(\beta_m y) \underbrace{\sinh(\gamma_{mn} z)}_{=\text{constant here}} * \sin(\alpha_p x) \sin(\beta_q y) \right) \right. \\ & \quad + \left(\sum_{m,n=0}^{\infty} B_{mn} \cos(\alpha_n x) \cos(\beta_m y) \underbrace{\sinh(\gamma_{mn} z)}_{=\text{constant here}} * \sin(\alpha_p x) \sin(\beta_q y) \right) \\ & \quad + \left(\sum_{m,n=0}^{\infty} C_{mn} \sin(\alpha_n x) \sin(\beta_m y) \underbrace{\cosh(\gamma_{mn} z)}_{=\text{constant here}} * \sin(\alpha_p x) \sin(\beta_q y) \right) \\ & \quad \left. + \left(\sum_{m,n=0}^{\infty} D_{mn} \cos(\alpha_n x) \cos(\beta_m y) \underbrace{\cosh(\gamma_{mn} z)}_{=\text{constant here}} * \sin(\alpha_p x) \sin(\beta_q y) \right) \right\} dx dy \end{aligned}$$

Fourier Functions: orthonormality properties of $\sin(\)$ and $\cos(\)$:

$$\int_0^{x_0} \sin(\alpha_n x) \sin(\alpha_p x) dx = \underbrace{\quad}_{\substack{\text{some} \\ \text{constant}}} \delta_{np} \begin{pmatrix} =1 & \text{for } n=p \\ =0 & \text{for } n \neq p \end{pmatrix}$$

$$\int_0^{x_0} \cos(\alpha_n x) \sin(\alpha_p x) dx = 0$$

Kroenecker δ -function: $\delta_{np} \begin{pmatrix} =1 & \text{for } n=p \\ =0 & \text{for } n \neq p \end{pmatrix}$

So all terms in above Σ 's vanish, except for a single term (in each sum) – that for the $A_{pq}/B_{pq}/C_{pq}/D_{pq}$ coefficient!!! The BC's will e.g. kill off 3 out of remaining 4 non-zero terms, thus only one term survives...

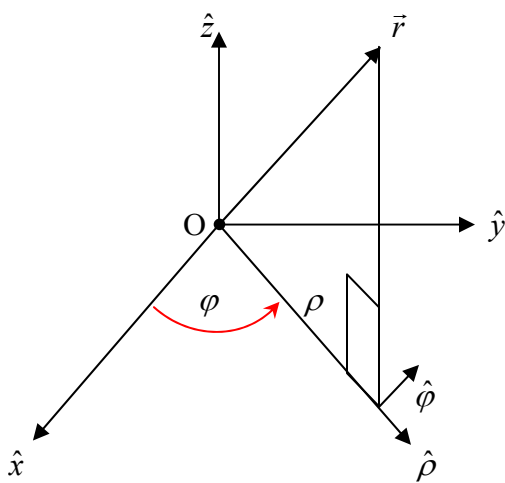
Suppose only the A_{pq} coefficient survives. Its analytic form is now known for all integers p and q .

Then the analytic form of 3-D potential $V(x, y, z)$ is now known – it is an infinite series solution of the form:

$$V(x, y, z) = \sum_{m,n=0}^{\infty} A_{mn} \sin(\alpha_n x) \sin(\beta_m y) \sinh \left(\underbrace{\gamma_{mn} z}_{=\sqrt{\alpha_n^2 + \beta_m^2}} \right)$$

Laplace's Equation $\nabla^2 V(\rho, \varphi, z) = 0$

And Potential Problems with Cylindrical Symmetry (Cylindrical Coordinates)



$$\vec{r} = \vec{\rho} + \vec{z} = \rho \hat{\rho} + z \hat{z} \quad r = \sqrt{\rho^2 + z^2}$$

$$\begin{aligned} \nabla^2 V(\rho, \varphi, z) &= 0 \\ &= \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial V}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 V}{\partial \varphi^2} + \frac{\partial^2 V}{\partial z^2} = 0 \\ &= \frac{\partial^2 V}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial V}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 V}{\partial \varphi^2} + \frac{\partial^2 V}{\partial z^2} = 0 \end{aligned}$$

Again, we use the separation of variables technique:

$V(\rho, \varphi, z) = R(\rho)Q(\varphi)Z(z) \Rightarrow \nabla^2 V = 0 \Rightarrow$ yields 3 ordinary differential equations:

$$\left\{ \begin{aligned} \frac{d^2 Z(z)}{dz^2} - k^2 Z(z) &= 0 \Rightarrow Z(z) = e^{\pm kz} \\ \frac{d^2 Q(\varphi)}{d\varphi^2} + \nu^2 Q(\varphi) &= 0 \Rightarrow Q(\varphi) = e^{\pm i\nu\varphi} \\ \frac{d^2 R(\rho)}{d\rho^2} + \frac{1}{\rho} \frac{dR(\rho)}{d\rho} + \left(k^2 - \frac{\nu^2}{\rho^2} \right) R(\rho) &= 0 \end{aligned} \right.$$

Note(s):

- 1.) k is arbitrary without imposing boundary conditions.
- 2.) k appears in both $Z(z)$ and $R(\rho)$ equations.
- 3.) In order for $Q(\varphi)$ to be single-valued (i.e. $Q(\varphi) = Q(\varphi + 2\pi)$), ν must be an integer!

Let $x \equiv k\rho$ Then: $\frac{d^2 R(x)}{dx^2} + \frac{1}{x} \frac{dR(x)}{dx} + \left(1 - \frac{\nu^2}{x^2} \right) R(x) = 0 \Leftarrow$ Bessel's Equation

$$R(x) = x^\alpha \sum_{j=0}^{\infty} a_j x^j \Leftarrow \text{Power Series Solution} \quad \alpha = \pm \nu$$

$$a_{2j} \equiv -\frac{1}{4j(j+\alpha)} a_{2j-2} \quad \text{for } j = 0, 1, 2, 3, \dots$$

All odd powers of x_j have vanishing coefficients, i.e. $a_1 = a_3 = a_5 = a_{2j+1} = 0$

Coefficients a_{2j} expressed in terms of a_0 :

$$a_{2j} = \left[\frac{(-1)^j \Gamma(\alpha + 1)}{2^{2j} j! \Gamma(j + \alpha + 1)} \right] a_0 = \frac{(-1)^j}{2^{2(j+\alpha)} j! \Gamma(j + \alpha + 1)}$$

where $a_0 = \frac{1}{2^\alpha \Gamma(\alpha + 1)}$ $\Gamma(x) = \text{Gamma Function}$

There exist TWO solutions of the Radial Equation (i.e. Bessel's Equation):

They are:

Bessel Functions of 1st kind, of order $\pm\nu$:

$$\left. \begin{aligned} J_{+\nu}(x) &= \left(\frac{x}{2}\right)^\nu \sum_{j=0}^{\infty} \frac{(-1)^j}{j! \Gamma(j + \nu + 1)} \left(\frac{x}{2}\right)^{2j} \\ J_{-\nu}(x) &= \left(\frac{x}{2}\right)^{-\nu} \sum_{j=0}^{\infty} \frac{(-1)^j}{j! \Gamma(j - \nu + 1)} \left(\frac{x}{2}\right)^{2j} \end{aligned} \right\} \begin{array}{l} \text{These series converge for} \\ \text{all values of } x. \end{array}$$

n.b. $\left\{ \begin{array}{l} \text{If } \nu \text{ is not an integer (which is not the case here), then the } J_{\pm\nu}(x) \text{ form a pair of} \\ \text{linearly independent solutions to the 2nd order Bessel's Equation:} \\ \quad R(x) = A_\nu J_\nu(x) + A_{-\nu} J_{-\nu}(x) \text{ for } \nu \neq \text{integer} \end{array} \right.$

However, note that if $\nu = \text{integer}$ (which is the case for us here) then the Bessel functions $J_\nu(x)$ and $J_{-\nu}(x)$ are NOT linearly independent!!

If $\nu = m = \text{integer}$ (0, 1, 2, 3, ...), then $J_{-m}(x) = (-1)^m J_m(x)$

$\therefore \Rightarrow$ We must find another linearly independent solution for $R(x)$ when $\nu = m = \text{integer}$

It is "customary" to replace $J_{\pm\nu}(x)$ by just $J_\nu(x)$ and another function $N_\nu(x)$ (called Neumann Functions)

Where: $N_\nu(x) = \text{Bessel Function of 2nd kind} \equiv \frac{J_\nu(x) \cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)}$

NOTE: $N_\nu(x)$ is divergent (i.e singular) at $x \rightarrow 0$

Complex Bessel Functions = Bessel Functions of 3rd kind = Hankel Functions

Hankel Functions are complex linear combinations of $J_\nu(x)$ and $N_\nu(x)$ (Bessel Functions of 1st and 2nd kind respectively). They are defined as follows:

$\left. \begin{aligned} H_\nu^{(1)}(x) &\equiv J_\nu(x) + iN_\nu(x) \\ H_\nu^{(2)}(x) &\equiv J_\nu(x) - iN_\nu(x) \end{aligned} \right\} \begin{array}{l} \text{The Hankel Functions } H_\nu^{(1)}(x) \text{ and } H_\nu^{(2)}(x) \text{ also form a} \\ \text{fundamental set/basis of solutions to the Bessel equation.} \end{array}$

The General Solution for $\nabla^2 V(\rho, \varphi, z) = 0$ in Cylindrical Coordinates:

$$V(\rho, \varphi, z) = R(\rho)Q(\varphi)Z(z)$$

$$V(\rho, \varphi, z) = \sum_{m,n=0}^{\infty} J_m(k_{mn}\rho) \sinh(k_{mn}z) [A_{mn} \sin(m\varphi) + B_{mn} \cos(m\varphi)]$$

\swarrow $\cosh(k_{mn}z)$ is also allowed

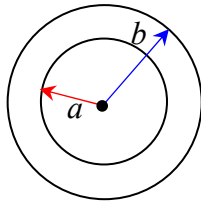
$$+ \sum_{m,n=0}^{\infty} N_m(k_{mn}\rho) \sinh(k_{mn}z) [C_{mn} \sin(m\varphi) + D_{mn} \cos(m\varphi)]$$

\swarrow $\cosh(k_{mn}z)$ is also allowed

Apply ALL boundary conditions on surfaces
 (and also impose for $r = \infty$, that $V(r = \infty) = \text{finite!}$ {If $r = \infty$ is part of the problem!})

Note that sometimes we want $V(\vec{r})$ only inside some finite region of space, e.g. coaxial capacitor – if so, then don't have to worry about $r = \infty$ solutions being finite – an example – the Coaxial Capacitor:

End View of a Coaxial Capacitor

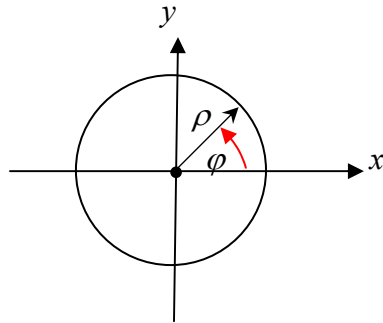


If the $\vec{r} = 0$ region is an excluded region in the problem, then must include (i.e. allow) the $N_\nu(x)$ solutions (singular at $x = k\rho = 0$)!!!

If $\vec{r} = 0$ region is included in problem, then ALL coefficients $C_{mn} = D_{mn} \equiv 0$ (for all m, n), if $V(\rho, \varphi, z)$ is finite @ $\vec{r} = 0$.

Using/imposing BC's on surfaces, orthogonality conditions on sines, cosines, $J_\nu(x)$, $N_\nu(x)$, etc. can find / determine values for all A_{mn} , B_{mn} , C_{mn} , D_{mn} coefficients!!

2-Dimensional Circular Symmetry
Laplace's Equation in (Circular) Cylindrical Coordinates



$$\nabla^2 V(\rho, \varphi) = 0$$

$$V(\rho, \varphi) = R(\rho)Q(\varphi) \leftarrow \text{Again try product solution}$$

Potential, $V(\rho, \varphi)$ is independent of z
 (e.g. infinitely long coaxial cable)

$$\nabla^2 V(\rho, \varphi) = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial V}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 V}{\partial \varphi^2} = 0$$

Get:

$$\frac{\rho}{R(\rho)} \frac{d}{d\rho} \left(\rho \frac{dR(\rho)}{d\rho} \right) = C_1 = -\frac{1}{Q(\varphi)} \frac{d^2 Q(\varphi)}{d\varphi^2} \quad \text{Let } C_1 = k^2$$

Then: $\rho \frac{d}{d\rho} \left(\rho \frac{dR(\rho)}{d\rho} \right) - k^2 R(\rho) = 0$ and $\frac{d^2 Q(\varphi)}{d\varphi^2} + k^2 Q(\varphi) = 0$

Require all solutions $Q(\varphi)$ to be single-valued, i.e. $Q(\varphi) = Q(\varphi + 2\pi)$
 because must have $V(\varphi) = V(\varphi + 2\pi)$.

Solutions for $Q(\varphi)$ are of the form:

$$Q(\varphi) = A \cos k\varphi + B \sin k\varphi$$

$$Q(\varphi) = Q(\varphi + 2k\pi) \text{ requires } k = \text{integer} = 0, \pm 1, \pm 2, \pm 3, \dots \pm n \dots$$

$$\frac{d^2 Q(\varphi)}{d\varphi^2} + n^2 Q(\varphi) = 0 \Rightarrow Q_n(\varphi) = A_n \cos(n\varphi) + B_n \sin(n\varphi)$$

$$\rho \frac{d}{d\rho} \left(\rho \frac{dR(\rho)}{d\rho} \right) - n^2 R(\rho) = 0 \Rightarrow R_n(\rho) = C_n \rho^n + D_n \rho^{-n} \quad \begin{matrix} \text{singular @ } \rho \rightarrow \infty \\ \text{singular @ } \rho = 0 \end{matrix} \text{ for } n \geq 1 \text{ (i.e. } n = 1, 2, 3, \dots)$$

$$R_o(\rho) = C_o + D_o \ln(\rho) \text{ for } n = 0 \text{ only}$$

General Solution for $\nabla^2 V(\rho, \varphi) = 0$ in Two Dimensions:
Cylindrical (a.k.a. Zonal) Harmonics

$$V(\rho, \varphi) = V_0 + V_1 \ln(\rho) + \sum_{n=1}^{\infty} \left[a_n \rho^n \cos(n\varphi) + b_n \rho^{-n} \cos(n\varphi) + c_n \rho^n \sin(n\varphi) + d_n \rho^{-n} \sin(n\varphi) \right]$$

Again, apply BC's on all relevant surfaces, impose $V(r \rightarrow \infty) = \text{finite}$, etc. – these will dictate / determine all coefficients, V_0 , V_1 , a_n , b_n , c_n and d_n .

i.e. Solve for V_0 , V_1 , a_n , b_n , c_n and d_n by applying all boundary conditions, $V(r \rightarrow \infty) = \text{finite}$, and using orthogonality conditions / properties:

$$a_n = "A" \int_0^{2\pi} d\varphi \int_0^{\rho_0} d\rho \rho V(\rho, \varphi) \rho^n \cos(n\varphi) \quad dA = \rho d\rho d\varphi$$

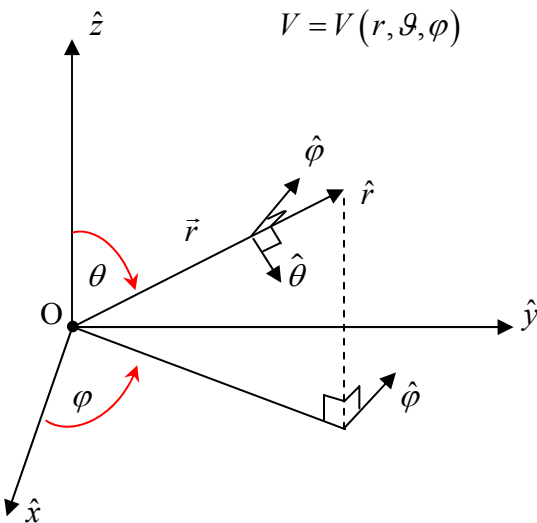
$$b_n = "B" \int_0^{2\pi} d\varphi \int_0^{\rho_0} d\rho \rho V(\rho, \varphi) \rho^{-n} \cos(n\varphi)$$

$$c_n = "C" \int_0^{2\pi} d\varphi \int_0^{\rho_0} d\rho \rho V(\rho, \varphi) \rho^n \sin(n\varphi)$$

$$d_n = "D" \int_0^{2\pi} d\varphi \int_0^{\rho_0} d\rho \rho V(\rho, \varphi) \rho^{-n} \sin(n\varphi)$$

“A”, “B”, “C”, “D” are appropriate normalization factors (we will discuss later).

Laplace's Equation $\nabla^2 V(r, \vartheta, \varphi) = 0$ In Spherical Coordinates



$$\begin{aligned} \nabla^2 V(r, \vartheta, \varphi) &= 0 \\ &= \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial V}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 V}{\partial \varphi^2} = 0 \end{aligned}$$

Again, try separation of variables / try product solution:

$$V(r, \vartheta, \varphi) = \frac{U(r)}{r} P(\theta) Q(\varphi) \Leftarrow \text{of this form!!}$$

$$P(\theta) Q(\varphi) \frac{d^2 U(r)}{dr^2} + \frac{U(r) Q(\varphi)}{r^2 \sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{dP(\theta)}{d\theta} \right) + \frac{U(r) P(\theta)}{r^2 \sin^2 \theta} \frac{d^2 Q(\varphi)}{d\varphi^2} = 0$$

Multiply by $r^2 \sin^2 \theta / U(r) P(\theta) Q(\varphi)$:

$$\underbrace{r^2 \sin^2 \theta \left[\frac{1}{U(r)} \frac{d^2 U(r)}{dr^2} + \frac{1}{r^2 \sin \theta} \frac{1}{P(\theta)} \frac{d}{d\theta} \left(\sin \theta \frac{dP(\theta)}{d\theta} \right) \right]}_{\text{function of } r+\theta \text{ only}} + \underbrace{\frac{1}{Q(\varphi)} \frac{d^2 Q(\varphi)}{d\varphi^2}}_{\text{function of } \varphi \text{ only}} = 0$$

Now: $\frac{1}{Q(\varphi)} \frac{d^2 Q(\varphi)}{d\varphi^2} = -m^2 \Rightarrow \frac{d^2 Q(\varphi)}{d\varphi^2} + m^2 Q(\varphi) = 0$

Solutions are of the form: $Q(\varphi) = e^{\pm im\varphi}$ where $m = \text{integer} = 0, 1, 2, 3, \dots$

Since $V(r, \vartheta, \varphi) = V(r, \vartheta, \varphi \pm 2\pi)$ i.e. $Q(\varphi) = Q(\varphi \pm 2\pi)$

Then: $Q(\varphi)$ must be single-valued!

Thus:

$$r^2 \sin^2 \theta \left[\frac{1}{U(r)} \frac{d^2 U(r)}{dr^2} + \frac{1}{r^2 \sin \theta} \frac{1}{P(\theta)} \frac{d}{d\theta} \left(\sin \theta \frac{dP(\theta)}{d\theta} \right) \right] = +m^2$$

$$\frac{1}{U(r)} \frac{d^2 U(r)}{dr^2} = -\frac{1}{r^2 \sin \theta} \frac{1}{P(\theta)} \frac{d}{d\theta} \left(\sin \theta \frac{dP(\theta)}{d\theta} \right) + \frac{m^2}{r^2 \sin^2 \theta}$$

multiply above equation by r^2 :

$$\underbrace{\frac{r^2}{U(r)} \frac{d^2 U(r)}{dr^2}}_{\text{function only of } r} = \underbrace{-\frac{1}{\sin \theta} \frac{1}{P(\theta)} \frac{d}{d\theta} \left(\sin \theta \frac{dP(\theta)}{d\theta} \right) + \frac{m^2}{\sin^2 \theta}}_{\text{function only of } \theta} = -\alpha \quad (\alpha \geq 0)$$

must hold for any/all r and θ !!

$$\therefore \frac{d^2 U(r)}{dr^2} - \frac{\alpha}{r^2} U(r) = 0 \quad \text{and} \quad \frac{1}{\sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{dP(\theta)}{d\theta} \right) + \left[\alpha - \frac{m^2}{\sin^2 \theta} \right] P(\theta) = 0$$

let / define $\alpha \equiv l(l+1)$ where $l = \text{integer} = 0, 1, 2, 3, \dots$

(Trust me, ☺ I know the answer . . .)

$$\therefore \frac{d^2 U(r)}{dr^2} - \frac{l(l+1)}{r^2} U(r) = 0 \quad \text{and} \quad \frac{1}{\sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{dP(\theta)}{d\theta} \right) + \left[l(l+1) - \frac{m^2}{\sin^2 \theta} \right] P(\theta) = 0$$

$$\begin{aligned} \text{Now let } x &= \cos \theta & x^2 &= \cos^2 \theta = 1 - \sin^2 \theta \\ dx &= -d \cos \theta = \sin \theta d\theta & \therefore \sin^2 \theta &= 1 - \cos^2 \theta = 1 - x^2 \\ & & \sin \theta &= \sqrt{1 - x^2} \end{aligned}$$

$$\underline{\text{Then:}} \quad \frac{1}{\sin \theta} \frac{d}{d\theta} \left(\frac{\sin^2 \theta}{\sin \theta} \frac{dP(\theta)}{d\theta} \right) + \left[l(l+1) - \frac{m^2}{\sin^2 \theta} \right] P(\theta) = 0$$

$$\underline{\text{Becomes:}} \quad \frac{d}{dx} \left((1-x^2) \frac{dP(x)}{dx} \right) + \left[l(l+1) - \frac{m^2}{(1-x^2)} \right] P(x) = 0 \quad \leftarrow \quad \text{Generalized Legendre' Equation}$$

General Solutions of the radial equation, $\frac{d^2 U(r)}{dr^2} - \frac{l(l+1)}{r^2} U(r) = 0$ are of the form:

$$U(r) = Ar^l + Br^{-(l+1)} \quad (l + A + B) \text{ are determined by boundary conditions...}$$

For $m = 0$ (azimuthally-symmetric problems – no ϕ -dependence) the general solution for azimuthally-symmetric potential $V(r, \theta)$ is of the form:

$$V(r, \theta) = \sum_{l=0}^{\infty} [A_l r^l + B_l r^{-(l+1)}] \underbrace{P_l(\cos \theta)}_{\substack{\text{"ordinary" Legendre' \\ Polynomial of order } l}}$$

The coefficients A_l and B_l are determined by the boundary conditions
 n.b. If \exists no charges at $r = 0$, then $B_l = 0 \quad \forall l !!$

Rodrigues' Formula is useful for “ordinary” Legendre' Polynomials:

$$P_l(x) \equiv \left(\frac{1}{2^l l!} \right) \frac{d^l}{dx^l} (x^2 - 1)^l$$

The coefficients A_l and B_l can be found / determined by evaluating $V(r, \theta)$ on the conducting surfaces in the problem, e.g. suppose we want to determine $V(\vec{r})$ inside a conducting sphere of radius $r = a$. Then on the surface of the conducting sphere at radius $r = a$ (an equipotential!):

$$V(r = a, \theta) = \sum A_l a^l P_l(\cos \theta) = \text{constant} \quad \leftarrow \quad \text{Legendre' Series}$$

n.b. inside conducting sphere, e.g. there are no charges at $r = 0 \quad \therefore B_l = 0 \quad \forall l$

In order to determine coefficients, take inner product:

$$A_l = \underbrace{\left(\frac{(2l+1)}{2a^l} \right)}_{\substack{\text{normalization} \\ \text{factor}}} \int_0^\pi V(r = a, \theta) P_l(\cos \theta) \sin \theta d\theta$$

Orthogonality condition on $P_l(x)$'s:

$$\int_{-1}^{-1} P_{l'}(x) P_l(x) dx = \frac{2}{(2l+1)} \underbrace{\delta_{l'l}}_{\substack{\text{Kronecker} \\ \delta\text{-function}}} \begin{cases} \delta_{l'l} = 0 \text{ for } l' \neq l \\ \delta_{l'l} = 1 \text{ for } l' = l \end{cases}$$

n.b. The $P_l(\cos \theta)$ functions form a complete orthonormal basis set on the unit circle ($r = 1$) for $-1 \leq \cos \theta \leq 1$ or: $0 \leq \theta \leq \pi$

“Ordinary” Legendre’ Polynomials $P_l(x)$ ($x = \cos \theta$) defined on the interval $-1 \leq x \leq 1$:

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

$$P_5(x) = \frac{1}{8}(63x^5 + 70x^3 + 15x)$$

...

Note: All $P_{l=\text{even}}(x)$ functions are even functions of x : $P_{l=\text{even}}(-x) = +P_{l=\text{even}}(x)$

All $P_{l=\text{odd}}(x)$ functions are odd functions of x : $P_{l=\text{odd}}(-x) = -P_{l=\text{odd}}(x)$

under $x \rightarrow -x$ reflection. Generally speaking, $P_l(-x) = (-1)^l P_l(x)$.

If 3-D spherical coordinate problem DOES have azimuthal / φ -dependence, then $m^2 \neq 0$ in Associated Legendre' Equation (A.L.E.):

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{dP(\theta)}{d\theta} \right) + \left[l(l+1) - \frac{m^2}{\sin^2 \theta} \right] P(\theta) = 0 \quad x = \cos \theta$$

Solutions to A.L.E. are Associated Legendre' Polynomials (A.L.P.'s)

Associated Legendre' Polynomials: $P_l^m(x) \equiv (-1)^m (1-x^2)^{\frac{m}{2}} \frac{d^m}{dx^m} \underbrace{P_l(x)}_{\substack{\text{"ordinary"} \\ \text{Legendre'} \\ \text{Polynomial}}}$

$m = \pm$ integer $\neq 0$

i.e. $m = \pm 1, \pm 2, \pm 3, \dots$

but have constraint on m !!!

$$\boxed{-l \leq m \leq +l}$$

i.e. $m = -l, -l+1, -l+2, \dots, -2, -1, 0, +1, +2, \dots, l-2, l-1, l$

Also: $P_l^{-m}(x) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(x)$

Orthogonality condition for $P_l^m(x)$ for fixed m :

$$\int_{-1}^1 P_l^m(x) P_l^m(x) dx = \frac{2}{(2l+1)} \frac{(l+m)!}{(l-m)!} \underbrace{\delta_{l'l}}_{\substack{\text{Kronecker} \\ \delta\text{-function}}}$$

We now define normalized $P(\theta)Q(\varphi)$ functions known as Spherical Harmonics:

$$Y_{lm}(\theta, \varphi) \equiv \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}} P_l^m(\cos\theta) \underbrace{e^{im\varphi}}_{=Q_m(\varphi)}$$

The Spherical Harmonics $Y_{lm}(\theta, \varphi)$ form a complete orthonormal set of basis “vectors” on the surface of the unit sphere ($r = 1$)

Note that $Y_{l-m}(\theta, \varphi) = (-1)^m Y_{lm}^*(\theta, \varphi)$ complex conjugate
i.e. $i \rightarrow -i$ where $i \equiv \sqrt{-1}$

$Y_{lm}(\theta, \varphi)$ Normalization and Orthogonality Condition:

$$\int_0^{2\pi} d\varphi \int_0^\pi \sin\theta d\theta Y_{l'm'}^*(\theta, \varphi) Y_{lm}(\theta, \varphi) = \delta_{l'l} \delta_{m'm}$$

i.e. $\int_{\Omega=0}^{\Omega=4\pi} d\Omega Y_{l'm'}^*(\theta, \varphi) Y_{lm}(\theta, \varphi) = \delta_{l'l} \delta_{m'm} \quad d\Omega = \sin\theta d\theta d\varphi$

Completeness' Relation: $\sum_{l=0}^{\infty} \sum_{m=-l}^l Y_{lm}^*(\theta, \varphi) Y_{lm}(\theta, \varphi) = \underbrace{\delta(\cos\theta - \cos\theta') \delta(\varphi - \varphi')}_{\text{DIRAC } \delta\text{-functions}}$

$Y_{lm}(\theta, \varphi)$ Spherical Harmonics

$$l=0 \quad \left\{ Y_{00} = \frac{1}{\sqrt{4\pi}} \right.$$

Use $Y_{l-m}(\theta, \varphi) = (-1)^m Y_{lm}^*(\theta, \varphi)$

in order to obtain $Y_{2-2}, Y_{2-1}, Y_{3-3}, Y_{3-2}, Y_{3-1}$ etc.

$$l=1 \quad \left\{ \begin{array}{l} Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin \theta e^{i\varphi} \\ Y_{10} = -\sqrt{\frac{3}{4\pi}} \cos \theta \end{array} \right.$$

Note:

$$Y_{lm}(\theta, \varphi) \equiv \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}} P_l^m(\cos \theta) e^{im\varphi}$$

$$l=2 \quad \left\{ \begin{array}{l} Y_{22} = \frac{1}{4} \sqrt{\frac{15}{2\pi}} \sin^2 \theta e^{2i\varphi} \\ Y_{21} = -\sqrt{\frac{5}{8\pi}} \sin \theta \cos \theta e^{i\varphi} \\ Y_{20} = \sqrt{\frac{5}{4\pi}} \left(\frac{3}{2} \cos^2 \theta - \frac{1}{2} \right) \end{array} \right.$$

$$Y_{l0}(\theta, \varphi) = \sqrt{\frac{(2l+1)}{4\pi}} P_l(\cos \theta)$$

$$l=3 \quad \left\{ \begin{array}{l} Y_{33} = -\frac{1}{4} \sqrt{\frac{35}{4\pi}} \sin^3 \theta e^{3i\varphi} \\ Y_{32} = \frac{1}{4} \sqrt{\frac{105}{2\pi}} \sin^2 \theta \cos \theta e^{2i\varphi} \\ Y_{31} = -\frac{1}{4} \sqrt{\frac{21}{4\pi}} \sin \theta (5 \cos^2 \theta - 1) e^{i\varphi} \\ Y_{30} = \sqrt{\frac{7}{4\pi}} \left(\frac{5}{2} \cos^3 \theta - \frac{3}{2} \cos \theta \right) \end{array} \right.$$

.... etc.

General Solution for Laplace's Equation $\nabla^2 V(r, \theta, \varphi)$ in Spherical Polar Coordinates

$$V(r, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} [A_{lm} r^l + B_{lm} r^{-(l+1)}] Y_{lm}(\theta, \varphi)$$

Coefficients A_{lm} and B_{lm} are determined by / from Boundary Conditions on Spherical Surface(s)

If $V = V(\theta, \varphi)$ on surface (e.g. at $r = a$)

(i.e. no charge at $r = 0$ in problem $\rightarrow B_{lm} = 0 \forall_{l,m}$)

Then: $V(\theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} A_{lm} Y_{lm}(\theta, \varphi)$ on surface ($r = a$).

And: $A_{lm} = \int_0^{4\pi} d\Omega Y_{lm}^*(\theta, \varphi) V(\theta, \varphi)$ on surface ($r = a$).

Note: $V(\underbrace{\theta=0}_{\text{"north pole"}}, \varphi) = \sum_{l=0}^{\infty} \sqrt{\frac{(2l+1)}{4\pi}} A_{l0}$

$$A_{l0} = \sqrt{\frac{(2l+1)}{4\pi}} \int_0^{4\pi} d\Omega P_l(\cos\theta) V(\theta, \varphi)$$

General Comments: The method of separation of variables used in Laplace's Equation $\nabla^2 V = 0$ in rectangular, cylindrical and spherical coordinates shows up again in Poisson's

Equation $\nabla^2 V = -\frac{\rho_{free}}{\epsilon_0}$ and also in the Wave Equation (valid for all classical wave phenomena)

$\nabla^2 \psi(\vec{r}, t) + \frac{1}{c^2} \frac{\partial^2 \psi(\vec{r}, t)}{\partial t^2} = 0$ and in Schrodinger's Wave Equation $H\psi = E\psi$ in Quantum

Mechanics problems. These equations will appear again and again, in one form or another for *E&M*, Classical Mechanics, Quantum Mechanics courses as well as for Classical / Newtonian Gravity problems...

For more detailed information e.g. on separation of variables and solutions to 3-D Wave

Equation $\nabla^2 \psi = -\frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2}$ in rectangular, cylindrical and spherical coordinates see S. Errede

lecture notes (Lecture IV – parts 1 & 2) on (sound) waves in 1-D, 2-D, 3D Physics 498 POM website:

http://online.physics.uiuc.edu/courses/phys498pom/498pom_lectures.html

and also see/read his Fourier Analysis Lectures on this website, if interested.