

## LECTURE NOTES 4

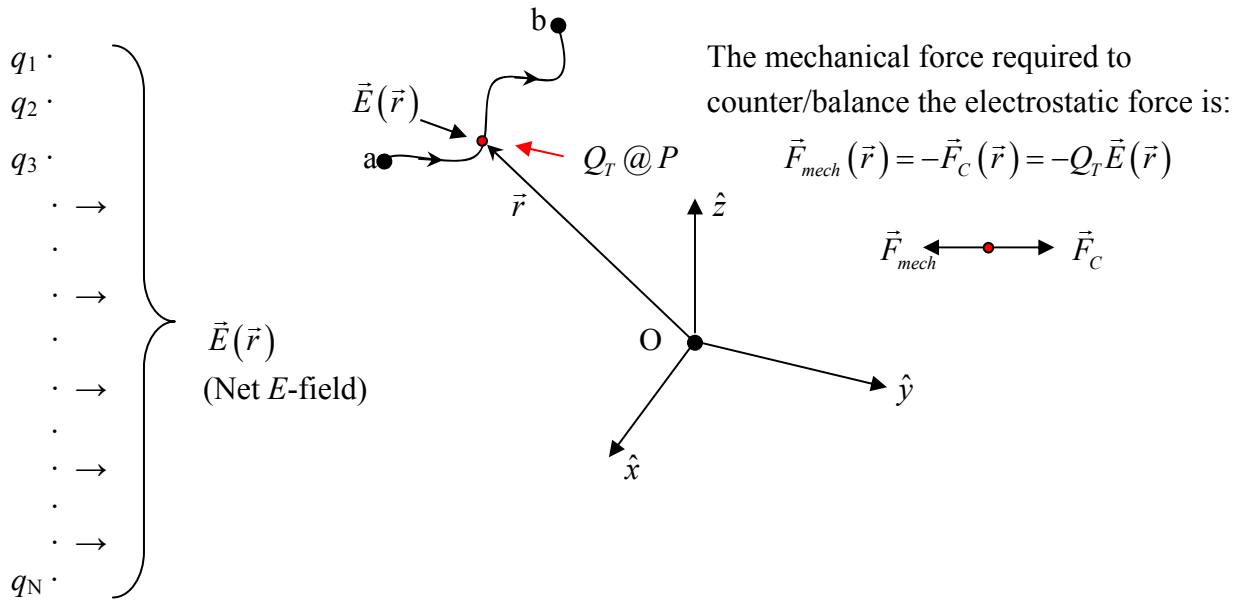
### Work & Electrostatic Energy:

Consider the following situation:

A stationary / fixed configuration of source charges is used to generate a (net) electric field  $\vec{E}(\vec{r})$ .

A test charge  $Q_T$  is moved from point  $a$  to point  $b$  in this electric field  $\vec{E}(\vec{r})$ . How much mechanical work  $W$  is done on the test charge  $Q_T$  in moving it (slowly) from point  $a$  to point  $b$ ?

At any point  $\vec{r}$  along the path  $a \rightarrow b$ , the electrostatic force acting on  $Q_T$  is:  $\vec{F}_C(\vec{r}) = Q_T \vec{E}(\vec{r})$



The mechanical work done on the test charge  $Q_T$  along the path  $a \rightarrow b$  is:

$$W = \int_a^b \vec{F}_{mech}(\vec{r}) \cdot d\vec{l} = -Q_T \int_a^b \vec{E}(\vec{r}) \cdot d\vec{l}$$

But:  $\vec{E}(\vec{r}) = -\nabla V(\vec{r})$  OR:  $\Delta V_{ab} \equiv V(b) - V(a) = -\int_a^b \vec{E}(\vec{r}) \cdot d\vec{l}$

$$\therefore W = Q_T [V(b) - V(a)] = Q_T \Delta V_{ab}$$

The mechanical work done on the test charge  $Q_T$  along the path  $a \rightarrow b$ :

$$\vec{F}_{mech}(\vec{r}) = -\vec{F}_C(\vec{r}) = -Q_T \vec{E}(\vec{r})$$

$$W = Q_T \Delta V_{ab} = Q_T [V(b) - V(a)] = \int_a^b \vec{F}_{mech}(\vec{r}) \cdot d\vec{l} = -\int_a^b \vec{F}_C(\vec{r}) \cdot d\vec{l} = -Q_T \int_a^b \vec{E}(\vec{r}) \cdot d\vec{l}$$

Now, if point  $a$  is the reference point  $\vec{r}_a = \infty$  where  $V(\vec{r}_a) = V(\infty) = 0$  and point  $\vec{r}_b = \vec{r}$

$$\text{Then: } W = Q_T \left[ V(\vec{r}) - \cancel{V(\infty)} \right] = Q_T V(\vec{r})$$

i.e. the test charge  $Q_T$  is (slowly) mechanically moved from the point  $\vec{r}_a = \infty$  to the point  $\vec{r}_b = \vec{r}$ .

Then  $W = Q_T V(\vec{r})$  is the mechanical work done on the charge  $Q_T$  in moving it along the path  $a \rightarrow b$  from  $\vec{r}_a = \infty$  to  $\vec{r}_b = \vec{r}$ .

Thus, now we also see that  $W =$  work done on charge  $Q_T$  is also equal to the POTENTIAL ENERGY,  $P.E. = W = Q_T V(\vec{r})$   $P.E.$  is linearly proportional to the potential  $V(r)$  (when referenced to  $V(r_a = \infty)$ ).

POTENTIAL ENERGY = amount of work  $W$  it takes to create the system (Joules).

POTENTIAL DIFFERENCE = work  $W$  it takes to create the system per unit charge (Joules/Coulomb = Volts).

UNITS OF WORK,  $W =$  UNITS OF POTENTIAL ENERGY,  $P.E. =$  JOULES (SI UNITS)

$$W = P.E. = Q_T V(\vec{r}) = \text{Coulomb-Volts} = \text{Joules}$$

$$\text{i.e. } 1 \text{ Coulomb} \times 1 \text{ Volt} = 1 \text{ Joule}$$

$$\text{Fundamental unit of electric charge: } 1e = 1.602 \times 10^{-19} \text{ Coulombs} = 1.602 \times 10^{-19} \text{ C}$$

$$\therefore 1 \underbrace{\text{electron volt}}_{= eV} = 1.602 \times 10^{-19} \text{ Joules} \leftarrow \text{energy conversion factor for eV} \Leftrightarrow \text{Joules}$$

Example 1:

An electron is initially infinitely far away from a proton, and is initially at rest (as is the proton). Assume the proton is fixed / rigidly attached to the head of a pin at the origin  $O(x, y, z) = (0, 0, 0)$ .

The electron is “released” at zero velocity and is attracted to the proton. What is its kinetic energy and speed when it is a distance of 1 meter from the proton?

Use Energy Conservation:

$$E_{TOT}^{INIT} = E_{TOT}^{FINAL} = 0$$

$$E_{TOT}^{INIT} = KE_e^{INIT} + PE_e^{INIT} = 0$$

$$E_{TOT}^{FINAL} = KE_e^{FINAL} + PE_e^{FINAL} = 0$$

$$KE_e^{INIT} = \frac{1}{2} m_e v_{init}^2 = 0$$

$$KE_e^{FINAL} = \frac{1}{2} m_e v_{final}^2 \geq 0$$

$$v_{init} = 0$$

$$v_{final} = ??$$

$$\begin{aligned}
 Q_e = -e \quad PE_e^{INIT} = Q_e V_p(\vec{r}_a = \infty) & \quad PE_e^{FINAL} = Q_e V_p(\vec{r}_b = 1m) \quad \text{where } V_p(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{Q_p}{r}, \\
 & \quad = -eV_p(\vec{r}_a = \infty) & \quad = -eV_p(\vec{r}_b = 1m) & \quad \text{with } Q_p = +e. \\
 & \quad = -\frac{1}{4\pi\epsilon_0} \frac{e^2}{\infty} = 0 & \quad = -\frac{1}{4\pi\epsilon_0} \left( \frac{e^2}{1} \right) < 0
 \end{aligned}$$

$$\text{So: } E_{TOT}^{INIT} = 0 = E_{TOT}^{FINAL} = KE_e^{FINAL} + PE_e^{FINAL} = 0$$

$$\text{Or: } KE_e^{FINAL} = -PE_e^{FINAL} = +\frac{1}{4\pi\epsilon_0} \left( \frac{e^2}{1} \right)$$

$r = 1 \text{ meter}$

$$e = 1.602 \times 10^{-19} \text{ coulomb}$$

$$\epsilon_0 = 8.85 \times 10^{-12} \text{ Farads/meter}$$

$$\therefore KE_e^{FINAL} = \frac{1}{4\pi * 8.85 \times 10^{-12}} \frac{(1.6 \times 10^{-19})^2}{1} \text{ Joules}$$

$$KE_e^{FINAL} = 2.3 \times 10^{-28} \text{ Joules}$$

$$\text{But: } \boxed{1 eV = 1.6 \times 10^{-19} \text{ Joules}} \quad \text{OR: } 1 \text{ Joule} = \frac{1 eV}{1.6 \times 10^{-19}}$$

$$\boxed{1 \text{ Joule} = 6.242 \times 10^{18} eV}$$

$$\therefore \underline{\underline{KE_e^{FINAL} = 1.437 \times 10^{-9} eV}}$$

$$\text{Now } KE_e^{FINAL} = \frac{1}{2} m_e v_e^{2FINAL} = 2.3 \times 10^{-28} \text{ Joules}$$

$$\text{But } m_e = 9.11 \times 10^{-31} \text{ kg}$$

$$1 \text{ Joule} = 1 \text{ Newton-meter}$$

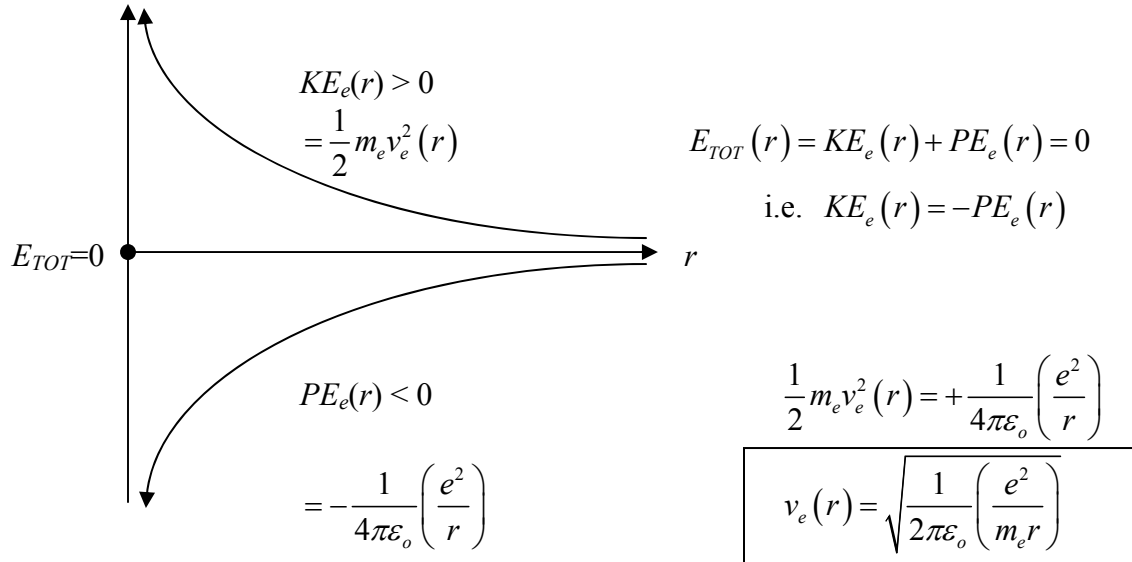
$$W = \int_a^b \vec{F}_{mech} \cdot d\vec{l}$$

$$v_e^{FINAL} = \sqrt{\frac{2KE_e^{FINAL}}{m_e}} = \sqrt{\frac{2 * 2.3 \times 10^{-28} \text{ Joules}}{9.11 \times 10^{-31} \text{ kg}}} = \sqrt{504.94 (m/s)^2} \approx 22.5 \text{ m/s}$$

$$v_e^{FINAL} \approx 22.5 \text{ m/sec} \quad \text{at} \quad r = 1m$$

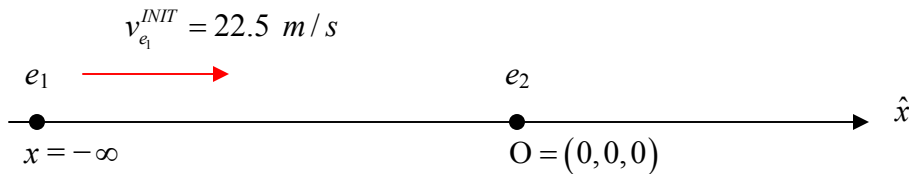
$$\text{NOTE: } v_e^{FINAL} (r = 1m) \ll c \quad \left. \begin{array}{l} \text{speed} \\ \text{of light} \end{array} \right\}$$

$$(22.5 \text{ m/s}) \quad (c = 3 \times 10^8 \text{ m/s})$$



Example 2:

An electron is located at  $x = -\infty$  and has an initial velocity  $\vec{v} = 22.5 \text{ m/s } \hat{x}$  and is initially infinitely far away from another electron, which is rigidly fixed at the origin  $O(x, y, z) = (0, 0, 0)$  at rest.



What is the distance of closest approach of  $e_1$  to  $e_2$ ? Why is there a minimum distance?

We know that  $E_{e_1}^{TOT\ INIT} = E_{e_1}^{TOT\ FINAL}$  (Energy is conserved in this process)

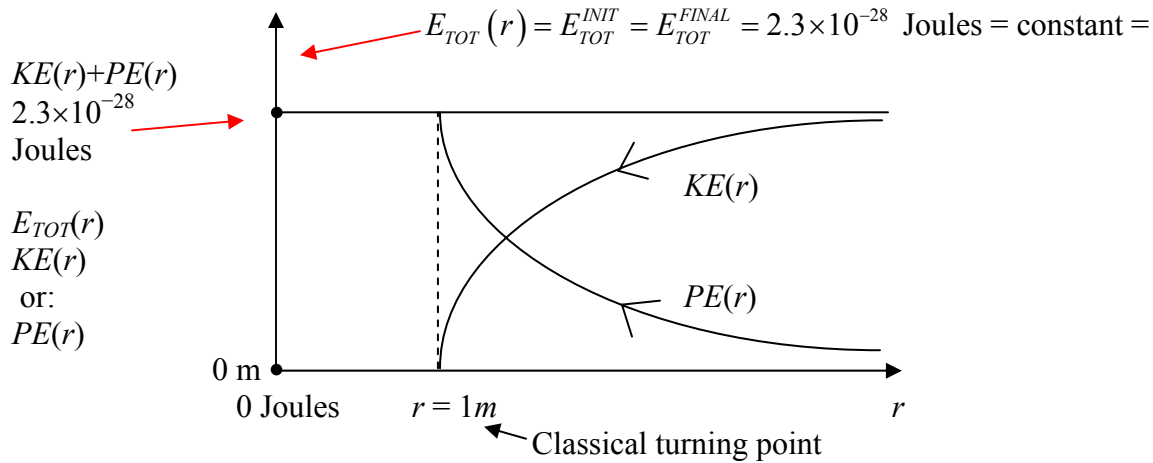
$$E_{e_1}^{TOT\ INIT} = KE_{e_1}^{INIT} + PE_{e_1}^{INIT} \qquad E_{e_1}^{TOT\ FINAL} = KE_{e_1}^{FINAL} + PE_{e_1}^{FINAL}$$

$$= \frac{1}{2} m_e v_{INIT}^2 + 0 = 2.3 \times 10^{-28} \text{ Joules} \qquad = 0 + \frac{1}{4\pi\epsilon_0} \frac{e^2}{r^2}$$

↙ 22.5 m/s
↖ repulsive
↗ closest approach

Solve for  $r_{closest\ approach}$  :

$$\therefore r_{closest\ approach} = \sqrt{\frac{1}{2\pi\epsilon_0} \left( \frac{e^2}{m_e v_{init}^2} \right)} = \sqrt{\frac{1}{2\pi * 8.85 \times 10^{-12}} \frac{(1.6 \times 10^{-19})^2}{9.11 \times 10^{-31} \times (22.5)^2}} = \underline{1 \text{ meter!}}$$



### What happens:

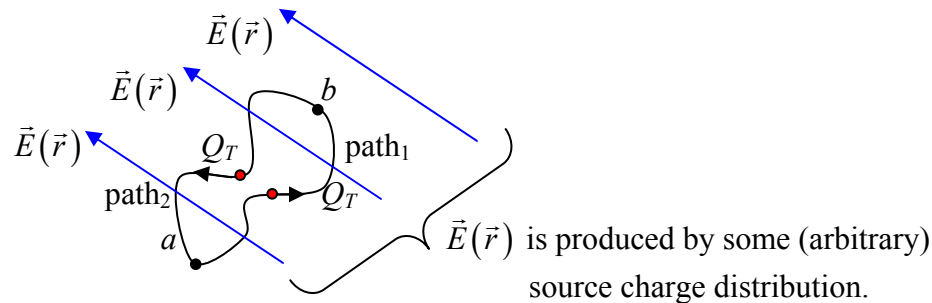
$e_1$  comes in from  $r = \infty$ , is decelerated and stopped at  $r = 1m$  (= distance of closest approach), and then  $e_1$  is accelerated back out to  $r = \infty$ .  $v_{e_1}^{final} = 22.5$  m/s at  $r = \infty$  again.

How much mechanical work is done in taking a charge  $Q_T$  around a closed path / closed contour  $C$  in an external electric field  $\vec{E}(\vec{r})$ ?

$$\begin{aligned}
 W &= \oint_C \vec{F}_{mech}(\vec{r}) \cdot d\vec{l} = \oint_C (-\vec{F}_C(\vec{r})) \cdot d\vec{l} = -\oint_C \vec{F}_C(\vec{r}) \cdot d\vec{l} \\
 &= -Q_T \oint_C \vec{E}(\vec{r}) \cdot d\vec{l} = 0 \quad \text{because:} \quad \oint_C \vec{E}(\vec{r}) \cdot d\vec{l} = 0
 \end{aligned}$$

NO work is done on closed contour, because the electrostatic Coulomb Force,  $\vec{F}_C(\vec{r})$  is a conservative force.

$\therefore$  Mechanical work done on test charge  $Q_T$  is independent of the path:



Note:  $\oint_C \vec{E}(\vec{r}) \cdot d\vec{l} = \int_a^b \vec{E}(\vec{r}) \cdot d\vec{l} + \int_b^a \vec{E}(\vec{r}) \cdot d\vec{l}$

path<sub>1</sub> path<sub>2</sub>

path<sub>1</sub> + path<sub>2</sub> are arbitrary because  $\vec{F}_C(\vec{r})$  is conservative (Coulomb Pot'l is  $1/r$ , central pot'l - no  $\theta, \varphi$  dependence.)

$$\vec{F}_C(\vec{r}) = -Q_T \nabla V(\vec{r})$$

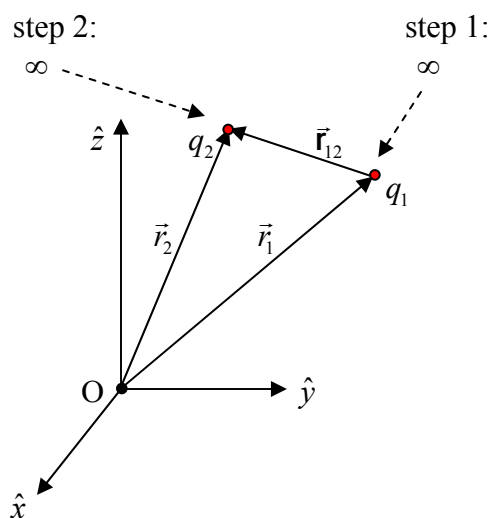
**ELECTROSTATIC ENERGY OF ASSEMBLY  
OF A POINT CHARGE DISTRIBUTION**

How much work does it take to assemble a collection of point charges – bringing them in from infinity, one by one? Bringing in the first charge  $q_1$  takes NO work ( $W_1=0$ ), since there is no electric field present, initially.

Now bring in  $2^{nd}$  charge  $q_2$ . The work done in bringing in  $q_2$  is:

$$W_2 = \frac{q_2}{4\pi\epsilon_0} \left( \frac{q_1}{r_{12}} \right) = q_2 V_1(\vec{r}_2) \quad V_1(\vec{r}_2) = \frac{1}{4\pi\epsilon_0} \left( \frac{q_1}{r_{12}} \right)$$

$\vec{r}_{12} = \vec{r}_2 - \vec{r}_1$        $r_{12} = |\vec{r}_{12}| = |\vec{r}_2 - \vec{r}_1| =$  separation distance between  $\vec{r}_2$  &  $\vec{r}_1$  (i.e. between charges  $q_1$  &  $q_2$ )



Now bring in  $3^{rd}$  charge. This requires work  $W_3 = q_3 V_{1,2}(\vec{r}_3)$  where  $V_{1,2}(\vec{r}_3)$  is the (total) potential due to charges  $q_1$  &  $q_2$ :

$$V_{1,2}(\vec{r}_3) = V_1(\vec{r}_3) + V_2(\vec{r}_3) = \frac{1}{4\pi\epsilon_0} \left( \frac{q_1}{r_{13}} + \frac{q_2}{r_{23}} \right)$$

$$\therefore W_3 = q_3 V_{1,2}(\vec{r}_3) = \frac{q_3}{4\pi\epsilon_0} \left( \frac{q_1}{r_{13}} + \frac{q_2}{r_{23}} \right)$$

Similarly for the  $4^{th}$  charge:  $V_{1,2,3}(\vec{r}_4) = V_1(\vec{r}_4) + V_2(\vec{r}_4) + V_3(\vec{r}_4) = \frac{1}{4\pi\epsilon_0} \left( \frac{q_1}{r_{14}} + \frac{q_2}{r_{24}} + \frac{q_3}{r_{34}} \right)$

$$\therefore W_4 = q_4 V_{1,2,3}(\vec{r}_4) = \frac{q_4}{4\pi\epsilon_0} \left( \frac{q_1}{r_{14}} + \frac{q_2}{r_{24}} + \frac{q_3}{r_{34}} \right)$$

The total work necessary to assemble the first 4 charges is thus:

$$\begin{aligned}
 W_{TOT} &= W_1 + W_2 + W_3 + W_4 = 0 + q_2 V_1(\vec{r}_2) + q_3 V_{1,2}(\vec{r}_3) + q_4 V_{1,2,3}(\vec{r}_4) \\
 &= \frac{1}{4\pi\epsilon_o} \left( \frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_1 q_4}{r_{14}} + \frac{q_2 q_3}{r_{23}} + \frac{q_2 q_4}{r_{24}} + \frac{q_3 q_4}{r_{34}} \right)
 \end{aligned}$$

We can generalize this for  $N$  charges:

$$W_{TOT} = \frac{1}{4\pi\epsilon_o} \sum_{i=1}^N \sum_{\substack{j=1 \\ j \neq i \\ j > i}}^N \left( \frac{q_i q_j}{r_{ij}} \right) = \frac{1}{8\pi\epsilon_o} \sum_{i=1}^N \sum_{\substack{j=1 \\ j \neq i}}^N \left( \frac{q_i q_j}{r_{ij}} \right)$$

so we don't count same pair twice!
double-counts pairs - but factor of 8 (vs. 4) takes care of this!

$$W_{TOT} = \frac{1}{2} \sum_{i=1}^N q_i \left( \underbrace{\sum_{\substack{j=1 \\ j \neq i}}^N \frac{1}{4\pi\epsilon_o} \left( \frac{q_j}{r_{ij}} \right)}_{=V(\vec{r}_i)} \right) = \frac{1}{2} \sum_{i=1}^N q_i V(\vec{r}_i)$$

Notice  $W_{TOT}$  can be  $< 0$  or can be  $> 0$ , depending on signs of  $q_i$  &  $q_j$

e.g.    if all  $q_i q_j$  terms  $> 0$ ,  $W_{TOT} > 0$  (for pair-wise repulsive forces)  
          if all  $q_i q_j$  terms  $< 0$ ,  $W_{TOT} < 0$  (for pair-wise attractive forces)

**ELECTROSTATIC ENERGY OF A  
CONTINUOUS CHARGE DISTRIBUTION**

For a discrete / discretized charge distribution:

$$W_{TOT} = \frac{1}{2} \sum_{i=1}^N q_i V(\vec{r}_i) = \frac{1}{2} \sum_{i=1}^N q_i \left( \sum_{\substack{j=1 \\ j \neq i}}^N \frac{1}{4\pi\epsilon_o} \left( \frac{q_j}{r_{ij}} \right) \right)$$

For a continuum volume charge density,  $\rho(\vec{r})$ :     $\sum_{i=1}^N q_i V(\vec{r}_i) \Rightarrow \int_V dq V(\vec{r}) = \int_V \rho(\vec{r}) V(\vec{r}) d\tau$

$$W_{TOT} = \frac{1}{2} \oint_V \rho(\vec{r}) V(\vec{r}) d\tau$$

For a continuum line charge density,  $\lambda(\vec{r})$ :

$$W_{TOT} = \frac{1}{2} \int_C \lambda(\vec{r}) V(\vec{r}) dl$$

For a continuum surface charge density,  $\sigma(\vec{r})$ :

$$W_{TOT} = \frac{1}{2} \int_S \sigma(\vec{r}) V(\vec{r}) dA$$

Using Gauss' Law:

$$\rho(r) = \epsilon_o \nabla \cdot \vec{E}$$

$$W_{TOT} = \frac{1}{2} \int_V \rho(\vec{r}) V(\vec{r}) d\tau = \frac{\epsilon_o}{2} \int_V (\nabla \cdot \vec{E}) V(\vec{r}) d\tau$$

Now use integration by parts to transfer the derivative from  $\vec{E}$  to  $V$  (and also use the divergence theorem):

$$\int_V \nabla \cdot (f \vec{A}) d\tau = \int_V f (\nabla \cdot \vec{A}) d\tau + \int_V \vec{A} \cdot (\nabla f) d\tau = \oint_S f \vec{A} \cdot d\vec{a} \quad (\text{see Griffith's eqn. 1.59, p. 37})$$

$$\text{Or: } \int_V f (\nabla \cdot \vec{A}) d\tau = - \int_V \vec{A} \cdot (\nabla f) d\tau + \oint_S f \vec{A} \cdot d\vec{a}$$

$$\begin{aligned} \text{Thus: } W_{TOT} &= \frac{\epsilon_o}{2} \left[ - \int_V \vec{E}(\vec{r}) \cdot \underbrace{(\nabla V(\vec{r}))}_{\vec{E}(\vec{r})} d\tau + \oint_S V(\vec{r}) \vec{E}(\vec{r}) \cdot d\vec{a} \right] \\ &= -\vec{E}(\vec{r}) \end{aligned}$$

$$\begin{aligned}
 W_{TOT} &= \frac{\epsilon_0}{2} \left[ -\int_V \vec{E}(\vec{r}) \cdot (\nabla V(\vec{r})) d\tau + \oint_S V(\vec{r}) \vec{E}(\vec{r}) \cdot d\vec{A} \right] \\
 &= \frac{\epsilon_0}{2} \left[ \int_V (\vec{E}(\vec{r}) \cdot \vec{E}(\vec{r})) d\tau + \oint_S V(r) \vec{E}(\vec{r}) \cdot d\vec{A} \right] \\
 &= \frac{\epsilon_0}{2} \left[ \int_V E^2(\vec{r}) d\tau + \oint_S V(r) \vec{E}(\vec{r}) \cdot d\vec{A} \right]
 \end{aligned}$$

Now integration volume  $V$  and surface of integration  $S$  are arbitrary, as long as all charges are enclosed by the volume  $V$  and surface  $S$ .

So we can take  $\left( \begin{smallmatrix} S \\ V \end{smallmatrix} \right) \rightarrow$  all space (i.e.  $\infty$  - volume) without any loss of generality.

Then surface integral,  $\oint_S V(r) \vec{E}(\vec{r}) \cdot d\vec{A}$  vanishes, because:  $\left. \begin{array}{l} V(r = \infty) = 0 \\ \vec{E}(r = \infty) = 0 \end{array} \right\}$

All space

$$\therefore W_{TOT} = \frac{\epsilon_0}{2} \int_{\substack{V \\ \text{all} \\ \text{space}}} E^2(\vec{r}) d\tau > 0 \quad (\text{always!})$$

Note that for a single point charge,  $\vec{E}(\vec{r}) = \frac{q}{4\pi\epsilon_0} \frac{1}{r^2} \hat{r}$

$$\text{Then: } W_{TOT}^q = \frac{\epsilon_0}{2} \frac{q^2}{(4\pi\epsilon_0)^2} \int_{\substack{V \\ \text{all} \\ \text{space}}} \frac{1}{r^4} r^2 dr \sin\theta d\theta d\phi = \frac{q^2}{8\pi\epsilon_0} \int_0^\infty \frac{1}{r^2} dr = \infty!!!$$

Electrostatic energy associated with a point electric charge is infinite (singular)!

The origin  $r = 0$  cannot truly be included here – this formula breaks down – because it mathematically describes only a classical physics – we know that quantum mechanics takes over at distance scales of:

$$r \leq \tilde{\lambda}_e = \frac{\hbar c}{m_e c^2} \approx \frac{197 \text{ MeV} - \text{fm}}{0.511} \approx 385.5 \text{ fm} \approx 385.5 \times 10^{-15} \text{ m}$$

$$\tilde{\lambda}_e = \lambda_e / 2\pi = \frac{\hbar c}{m_e c^2} = \text{“reduced” Compton wave length of electron, here } h = \text{Planck’s constant,}$$

$$\text{and } \hbar = \frac{h}{2\pi}, \hbar c \approx 197 \text{ MeV} - \text{fm} \quad (1 \text{ MeV} = 10^6 \text{ eV}), \text{ electron rest mass energy,}$$

$$m_e c^2 = 511 \text{ keV} = 0.511 \text{ MeV}$$

$$\text{And } 1 \text{ fm} = 1 \text{ femto} - \text{meter} = 1 \text{ fermi} = 10^{-15} \text{ m}$$

Actually, we know that for atoms & molecules, quantum mechanics takes over at distance scales of  $\sim$  Bohr radius ( $H$ -atom):  $a_0 = 0.53 \text{ \AA} = 0.53 \times 10^{-10} \text{ m} = 0.053 \text{ nm} \gg \tilde{\lambda}_e = 385.5 \text{ fm} \approx 385.5 \times 10^{-15} \text{ m}$

The classical formula for  $W_{TOT}^q$  breaks down / is invalid for point charges, for  $r \leq \tilde{\lambda}_e = \frac{\hbar c}{m_e c^2} \approx 385.5 \text{ fm}$

$$W_{TOT} = \frac{\epsilon_0}{2} \int_V \vec{E}(\vec{r}) \cdot \vec{E}(\vec{r}) d\tau \quad (\text{Joules})$$

$$= \frac{\epsilon_0}{2} \int_V E^2(\vec{r}) d\tau > 0 \quad (\text{always})$$

This formula works fine/is valid for continuous charge distributions:  
 $\lambda(\vec{r}), \sigma(\vec{r}), \rho(\vec{r})$

$\vec{E}(\vec{r})$  is total/net electric field arising from charge distribution,  $\rho(\vec{r})$ .

Define:  $u_E(\vec{r}) \equiv \frac{\epsilon_0}{2} (\vec{E}(\vec{r}) \cdot \vec{E}(\vec{r})) = \frac{\epsilon_0}{2} E^2(\vec{r}) = \frac{1}{2} \rho(\vec{r}) V(\vec{r}) = \text{electrostatic energy density} =$   
 Joules/meter<sup>3</sup>

$$W_{TOT} = \int_V u_E(\vec{r}) d\tau = \frac{\epsilon_0}{2} \int_V E^2(\vec{r}) d\tau$$

Physically, says that the energy density,  $u_E(\vec{r}) = \frac{\epsilon_0}{2} E^2(\vec{r})$  is associated entirely with the electric field,  $\vec{E}(\vec{r})$ .

$$= \frac{1}{2} \oint_V \rho(\vec{r}) V(\vec{r}) d\tau$$

Physically, says that the energy density,  $u_E(\vec{r}) = \frac{1}{2} \rho(\vec{r}) V(\vec{r})$  is due to product (i.e. mix) of volume charge density,  $\rho(\vec{r})$  and electrostatic potential,  $V(\vec{r})$ .

Electrostatic energy density:  $u_E(\vec{r}) = \frac{\epsilon_0}{2} E^2(\vec{r}) = \frac{1}{2} \rho(\vec{r}) V(\vec{r})$  (Joules/meter<sup>3</sup>)

## ELECTROSTATIC ENERGY AND THE SUPERPOSITION PRINCIPLE

Note that since  $W$  (work) /  $P.E.$  (potential energy) is quadratic in the electric field, i.e.

$$W = \frac{\epsilon_0}{2} \int_V E^2(\vec{r}) d\tau = \frac{\epsilon_0}{2} \int_V \vec{E}(\vec{r}) \cdot \vec{E}(\vec{r}) d\tau$$

then for example, if we double  $\vec{E}$ , i.e.  $\vec{E} \rightarrow 2\vec{E}$ , the work needed / done to assemble the now-doubled charge distribution / stored potential energy in the electrostatic field quadruples (i.e. increases 4x).

Thus, work done to assemble the charge distribution / potential energy stored in the electrostatic field of the charge distribution does NOT obey the principle of linear superposition.

This is not surprising, as there exist many other physical examples where the superposition principle does not hold – e.g. the superposition of two (or more) acoustic / sound waves, or e.g. the superposition of two (or more) light waves – sound/light overall intensities (proportional to square of overall/total amplitude) result from interference effects at the amplitude level.

Here, e.g. let  $\vec{E}_{TOT}(\vec{r}) = \vec{E}_1(\vec{r}) + \vec{E}_2(\vec{r})$

$$\begin{aligned}
 \text{Then: } W_{TOT} = P.E. &= \frac{\epsilon_0}{2} \int_V \left( \vec{E}_{TOT}(\vec{r}) \cdot \vec{E}_{TOT}(\vec{r}) \right) d\tau \\
 &= \frac{\epsilon_0}{2} \int_V \left( \vec{E}_1(\vec{r}) + \vec{E}_2(\vec{r}) \right) \cdot \left( \vec{E}_1(\vec{r}) + \vec{E}_2(\vec{r}) \right) d\tau \\
 &= \frac{\epsilon_0}{2} \int_V \left( E_1^2(\vec{r}) + 2\vec{E}_1(\vec{r}) \cdot \vec{E}_2(\vec{r}) + E_2^2(\vec{r}) \right) d\tau
 \end{aligned}$$

$$W_{TOT} = P.E. = W_1 + W_2 + \epsilon_0 \int_V \left( \vec{E}_1(\vec{r}) \cdot \vec{E}_2(\vec{r}) \right) d\tau$$

Clearly, the principle of linear superposition is **not** (always) obeyed for  $W_{TOT}$  – there exists an additional cross-term involving the product the two electric fields:  $\vec{E}_1(\vec{r}) \cdot \vec{E}_2(\vec{r})$ .