Math review

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1. series approximations

Taylor's Theorem

$$f(x) = f(x_0) + \frac{df}{dx}\bigg|_{x=x_0}(x - x_0) + \ldots + \frac{1}{n!}\frac{d^n f}{dx^n}\bigg|_{x=x_0}(x - x_0)^n + \ldots$$

This can be handy. (Recall: $n! = 1 \cdot 2 \cdot 3 \cdot \ldots [n-1] \cdot n$)

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Binomial approximation

Work out an approximation for $(1+x)^n$ when x is small:

Use $x_0 = 0$.

$$\frac{d}{dx}(1+x)^n \quad = \quad n(1+x)^{n-1}$$

$$\frac{d^2}{dx^2}(1+x)^n \quad = \quad \frac{d}{dx}\, n(1+x)^{n-1} \quad = \quad n(n-1)(1+x)^{n-2}$$

so

$$(1+x)^n = (1+x)^n\Big|_{x=0} + \left[n(1+x)^{n-1}\Big|_{x=0}\right](x-0) +$$

$$\left[\frac{1}{2!}n(n-1)(1+x)^{n-2}\Big|_{x=0}\right](x-0)^2 + \ldots$$

$$= \; 1+nx+\frac{n(n-1)}{2}x^2+\ldots$$

when $x<<1$ we end up with $(1+x)^n \approx 1+nx.$

This works when *n* is a fraction, too.

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sin(x), for x in radians and x close to zero

$$x_0=0$$

$$\frac{d\sin(x)}{dx}=\cos(x) \qquad \frac{d^2\sin(x)}{dx^2}=-\sin(x)$$

$$\frac{d\cos(x)}{dx}=-\sin(x) \qquad \frac{d^2\cos(x)}{dx^2}=-\cos(x)$$

$$\sin(x) = \sin(0) + x\frac{d\sin(x)}{dx}\Big|_{x=0} + \frac{x^2}{2!}\frac{d^2\sin(x)}{dx^2}\Big|_{x=0} + \frac{x^3}{3!}\frac{d^3\sin(x)}{dx^3}\Big|_{x=0} + \ldots$$

$$= \sin(0) + x\cos(0) - \frac{x^2}{2!}\sin(0) - \frac{x^3}{3!}\cos(0) + \ldots$$

$$= 0 + x + 0 - \frac{x^3}{3!} + \ldots \approx x \quad \text{for } x \ll 1$$

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cos(x), for x in radians and x close to zero

again, $x_0 = 0$

$$\cos(x) = \cos(0) + x\frac{d\cos(x)}{dx}\Big|_{x=0} + \frac{x^2}{2!}\frac{d^2\cos(x)}{dx^2}\Big|_{x=0} + \frac{x^3}{3!}\frac{d^3\cos(x)}{dx^3}\Big|_{x=0} + \ldots$$

$$= 1 + 0 - \frac{x^2}{2!} + 0 + \ldots \approx 1 - \frac{x^2}{2!} \quad \text{for } x \ll 1$$

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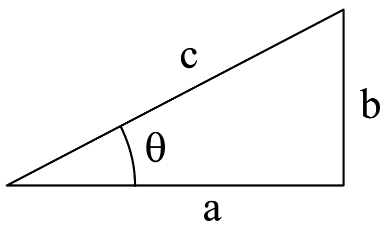
2. some geometry

right triangles



$a^2 + b^2 = c^2$

Integers n_1, n_2, n_3 for which $n_1^2 + n_2^2 = n_3^2$ are called Pythagorean triples.

Examples: 3, 4, 5; 5, 12, 13; 7, 24, 25; 8, 15, 17; 9, 40, 41; …

Trigonometric functions for right triangles:

$\sin\theta = b/c \qquad \cos\theta = a/c \qquad \tan\theta = b/a$
$\csc\theta = c/b \qquad \sec\theta = c/a \qquad \cot\theta = a/b$

Useful trig identities:

$\sin^2\theta + \cos^2\theta = 1 \qquad 1 + \cot^2\theta = \csc^2\theta \qquad 1 + \tan^2\theta = \sec^2\theta$

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$
$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$

$$\sin\alpha\sin\beta=\tfrac{1}{2}\cos(\alpha-\beta)-\tfrac{1}{2}\cos(\alpha+\beta)$$
$$\cos\alpha\cos\beta=\tfrac{1}{2}\cos(\alpha-\beta)+\tfrac{1}{2}\cos(\alpha+\beta)$$
$$\sin\alpha\cos\beta=\tfrac{1}{2}\sin(\alpha+\beta)+\tfrac{1}{2}\sin(\alpha-\beta)$$

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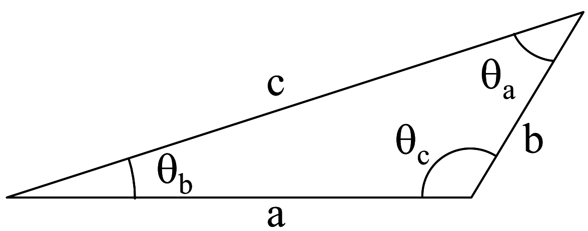
triangles in general



$$a^2 + b^2 - 2ab\cos\theta_c = c^2 \quad \text{(law of cosines)}$$

$$\frac{\sin(\theta_a)}{a}=\frac{\sin(\theta_b)}{b}=\frac{\sin(\theta_c)}{c} \quad \text{(law of sines)}$$

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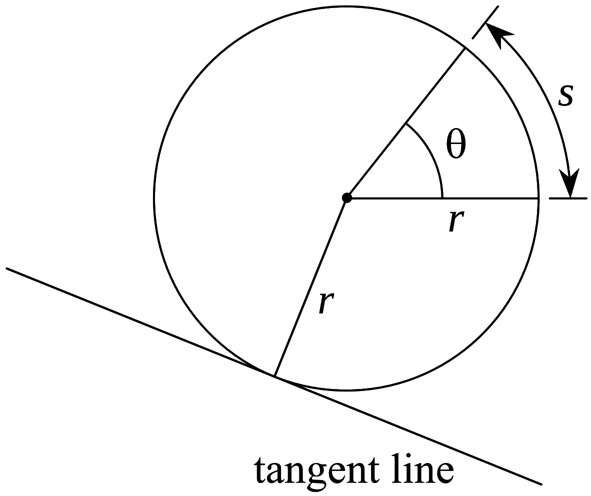
*circle*s



1. tangent and radius are perpendicular
2. $s = r\theta$ $\left(\theta \text{ in radians}\right)$

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3. exponentials

e^x takes off like a rocket for large x.

$$\frac{de^x}{dx} = e^x \quad \text{so} \quad \frac{d^n e^x}{dx^n} = e^x .$$

Taylor's theorem: $e^x = 1 + x + \dfrac{x^2}{2!} + \cdots + \dfrac{x^n}{n!} + \cdots$

This will be handy: for $i \equiv \sqrt{-1}$ (so that $i^2 = -1$),

$$e^{ix} = 1 + \left(ix\right) + \frac{\left(ix\right)^2}{2!} + \frac{\left(ix\right)^3}{3!} + \frac{\left(ix\right)^4}{4!} + \cdots$$

$$= 1 + ix \quad \frac{-x^2}{2!} \quad - i\frac{x^3}{3!} \quad + \frac{x^4}{4!} \quad + \cdots$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \cdots\right)$$

$$= \cos x + i \cdot \sin x .$$

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4. differential equations

It is common in physics to describe how quantities *change* in response to external circumstances. Because of this, calculus is the natural language for describing the physical world. Many of our statements about how things work are phrased as *differential equations.*

An example: consider a box full of radioactive atoms. The more atoms there are in the box, the more decays there should be per unit time.

In addition, the greater the decay rate (the shorter the half-life) for this species of atom, the more decays there'll be per unit time.

$N(t) \equiv$ number of surviving atoms inside the box at time *t.*

$\Gamma \equiv 1/\tau$ is the decay rate where τ is the mean life of an atom.

The number of decays per unit time is equal in magnitude, opposite in sign, to the change per unit time of the surviving population in the box.

Therefore, the statement . . .

"The number of decays per unit time equals the number of surviving atoms in the box times the decay rate for that species of atom"

is equivalent to the differential equation

$$-\frac{dN(t)}{dt} = N(t) \cdot \Gamma \quad \text{or} \quad \frac{dN(t)}{dt} = -\Gamma N(t) \ .$$

How can we solve this differential equation for $N(t)$?

Note that $\frac{dN(t)}{dt}$ is proportional to $N(t)$ since Γ is a constant.

From a few pages back, after replacing x with t:

$$\frac{de^t}{dt} = e^t \quad \text{so} \quad \frac{de^{at}}{dt} = ae^{at}$$

Note that $\frac{de^{at}}{dt}$ is proportional to e^{at} since a is a constant.

Compare: $\frac{dN(t)}{dt} = -\Gamma N(t)$ and $\frac{de^{at}}{dt} = ae^{at}$.

It looks like $N(t) \Leftrightarrow e^{at}$ and $a \Leftrightarrow -\Gamma$ works: $N(t) = e^{-\Gamma t}$ is a solution to the differential equation. But $N(t) = 2e^{-\Gamma t}$ is also a solution!

What to do? We have a first order differential equation, so we can determine the unique solution with the help of one initial condition. (A second order differential equation would require two initial conditions.)

If we know $N(0)$ we can use that: say there are N_0 atoms in the box at $t = 0$.

The only version of N(t) that solves the differential equation <u>and</u> satisfies the initial condition is $N(t) = N_0 e^{-\Gamma t}$.

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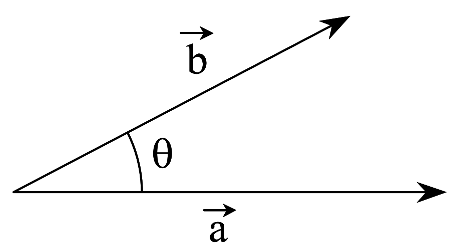
5. Vectors

Vectors have a length and a direction.



Unit vectors in cartesian coordinates: $\hat{x}, \hat{y}, \hat{z}$ or $\hat{i}, \hat{j}, \hat{k}$

Addition, subtraction, you know about.

Scalar product

$$\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}|\,\cos\theta.$$

Component of $\vec{a}$ along $\vec{b}$ is $|\vec{a}|\cos\theta = \left(\vec{a}\cdot\vec{b}\right)/|\vec{b}|$.

Also, if $\vec{a} = a_x\,\hat{x} + a_y\,\hat{y} + a_z\,\hat{z}$

$\vec{b} = b_x\,\hat{x} + \mathrm{b}_y\,\hat{y} + \mathrm{b}_z\,\hat{z}$

then

$$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z$$

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Cross product

$$\left|\vec{a} \times \vec{b}\right| = |\vec{a}|\,|\vec{b}|\,\sin\theta$$

Direction is given by right hand rule. For the vectors as drawn above:

$\vec{a} \times \vec{b}$ $\odot$ (out of the paper)

$\vec{b} \times \vec{a}$ $\otimes$ (into the paper)

$$\vec{a}\times\vec{b} = \left(a_x b_y - a_y b_x\right)\hat{z} + \left(a_y b_z - a_z b_y\right)\hat{x} + \left(a_z b_x - a_x b_z\right)\hat{y}.$$

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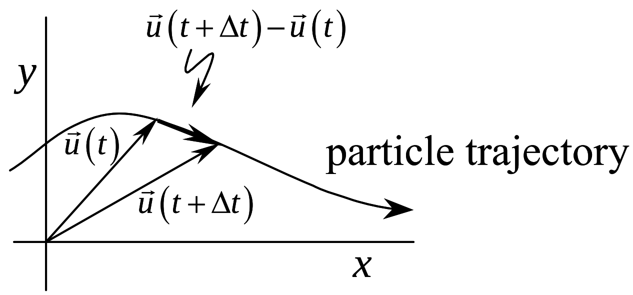
Differentiation of vectors

Go back to the basic definition of a derivative:

$$\frac{d}{dt}\vec{a}(t) \equiv \lim_{\Delta t \to 0} \frac{\vec{a}(t+\Delta t)-\vec{a}(t)}{\Delta t}$$



$\vec{v}(t) = \frac{d}{dt}\vec{x}(t)$ is an example where this is useful.

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Integration of vectors

Same idea: an integral is just a big sum.

$$\int_{t_1}^{t_2} \vec{a}(t)\, dt = \lim_{\Delta t \to 0} \left\{ \sum_{t_i = t_1}^{t_i = t_2} \vec{a}(t_i)\ \Delta t \right\}$$

For example, $\int_{t_1}^{t_2} \vec{v}(t)\, dt = \lim_{\Delta t \to 0} \left\{ \sum_{t_i = t_1}^{t_i = t_2} \vec{v}(t_i)\ \Delta t \right\}$

Recall that (for constant velocity) velocity × time = distance.

When Δt is small, $\vec{v}\left(t_i\right)\Delta t \Rightarrow$ displacement between t_i and $t_i + \Delta t$.

Therefore, $\int_{t_1}^{t_2} \vec{v}\left(t\right) dt$ is the net displacement between t_1 and t_2.

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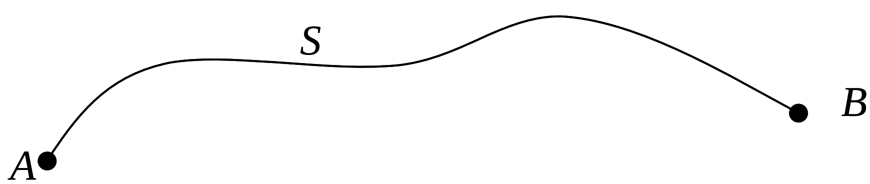
Line integrals



Travel from point *A* to point *B* along the path labeled *S*.

Say there's an applied force acting, which might be a function of position along the path *S*: $\vec{F}\left(\vec{s}\right)$.

Recall that the work done by the force through a small interval $\overrightarrow{\Delta s}$ is $\vec{F} \bullet \overrightarrow{\Delta s}$ as long as $\vec{F}$ is almost constant over the interval $\overrightarrow{\Delta s}$.

Net work done is $\sum_{\text{all intervals}} \vec{F}\left(\vec{s}_i\right) \bullet \overrightarrow{\Delta s_i}$.

In the limit that the intervals become infinitesimal, Net work = $\int_A^B \vec{F} \cdot d\vec{s}$.

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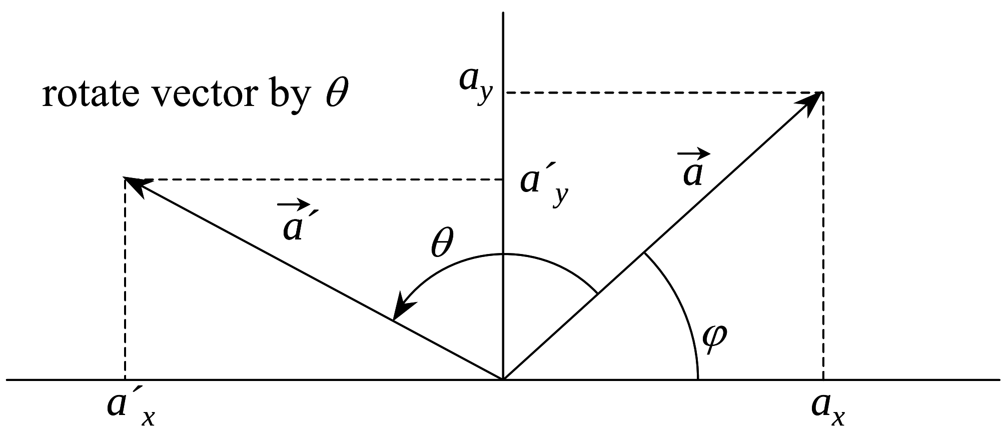
6. Vector transformation properties

Under rotations



let $a \equiv |\vec{a}|$. $a_x = a \cos\varphi$ $a_y = a \sin\varphi$

$a' = a$ since lengths of vectors are <u>invariant</u> under rotations.

$\therefore\ a'_x = a\cos(\theta+\varphi) \quad a'_y = a\sin(\theta+\varphi)$

useful trig identities:

$$\cos(\theta+\varphi) = \cos\theta\cos\varphi - \sin\theta\sin\varphi$$
$$\sin(\theta+\varphi) = \sin\theta\cos\varphi + \cos\theta\sin\varphi$$

use this to rewrite . . .

$$\begin{aligned} a'_x &= a\cos\varphi\cos\theta - a\sin\varphi\sin\theta \\ &= a_x\cos\theta - a_y\sin\theta \end{aligned}$$

$$\begin{aligned} a'_y &= a\sin\varphi\cos\theta + a\cos\varphi\sin\theta \\ &= a_y\cos\theta + a_x\sin\theta \end{aligned}$$

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Matrix multiplication and rotation matrices

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} u & v \\ w & x \end{bmatrix} = \begin{bmatrix} (au+bw) & (av+bx) \\ (cu+dw) & (cv+dx) \end{bmatrix}$$

also:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (ax+by) \\ (cx+dy) \end{bmatrix}$$

We can write $\vec{a}$ as $\begin{bmatrix} a_x \\ a_y \end{bmatrix}$, $\vec{a}'$ similarly to get . . .

$$\begin{bmatrix} a'_x \\ a'_y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a_x \\ a_y \end{bmatrix}$$

Note the utility of this: successive rotations can be represented as products of matrices…

$$\begin{bmatrix} a_x'' \\ a_y'' \end{bmatrix} = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix} \begin{bmatrix} a_x' \\ a_y' \end{bmatrix}$$

$$= \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix} \left\{ \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} a_x \\ a_y \end{bmatrix} \right\}$$

$$= \left\{ \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \right\} \begin{bmatrix} a_x \\ a_y \end{bmatrix}$$

since matrix multiplication is <u>associative</u>.

This works fine in 3 dimensions:

$$\begin{bmatrix} a_x' \\ a_y' \\ a_z' \end{bmatrix} = \underset{\sim}{R} \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}$$ where $\underset{\sim}{R}$ is 3 × 3 <u>rotation matrix</u>.

How unit vectors transform under rotations

$\hat{x} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$. In the rotated frame it becomes

$$\begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} R_{11} \\ R_{21} \\ R_{31} \end{bmatrix}$$

(Note that 1st index is row, 2nd is column).

So: the 1st column of $\underset{\sim}{R}$ is the same as the representation of $\hat{x}$ in the new frame after rotation.

Similarly, the 2nd column of $\underset{\sim}{R}$ is the same as the representation of $\hat{y}$ and the 3rd is $\hat{z}$.

$\hat{x}\cdot\hat{y}=0$, $\hat{y}\cdot\hat{z}=0$, $\hat{z}\cdot\hat{x}=0$, regardless of which coordinate system we use (scalar product is invariant under rotations) so each column of $\underset{\sim}{R}$ is perpendicular to every other column.

$$\hat{x}' \cdot \hat{y}' = 0, \quad \hat{y}' \cdot \hat{z}' = 0, \quad \hat{z}' \cdot \hat{x}' = 0 \qquad \text{so...}$$

$$\hat{x}' \cdot \hat{y}' = R_{11}R_{12} + R_{21}R_{22} + R_{31}R_{32} = 0$$

$$\hat{y}' \cdot \hat{z}' = R_{12}R_{13} + R_{22}R_{23} + R_{32}R_{33} = 0$$

$$\hat{z}' \cdot \hat{x}' = R_{13}R_{11} + R_{23}R_{21} + R_{33}R_{31} = 0$$

Transformations of this sort are called *orthogonal transformations*: quantities that are orthogonal before remain orthogonal after.

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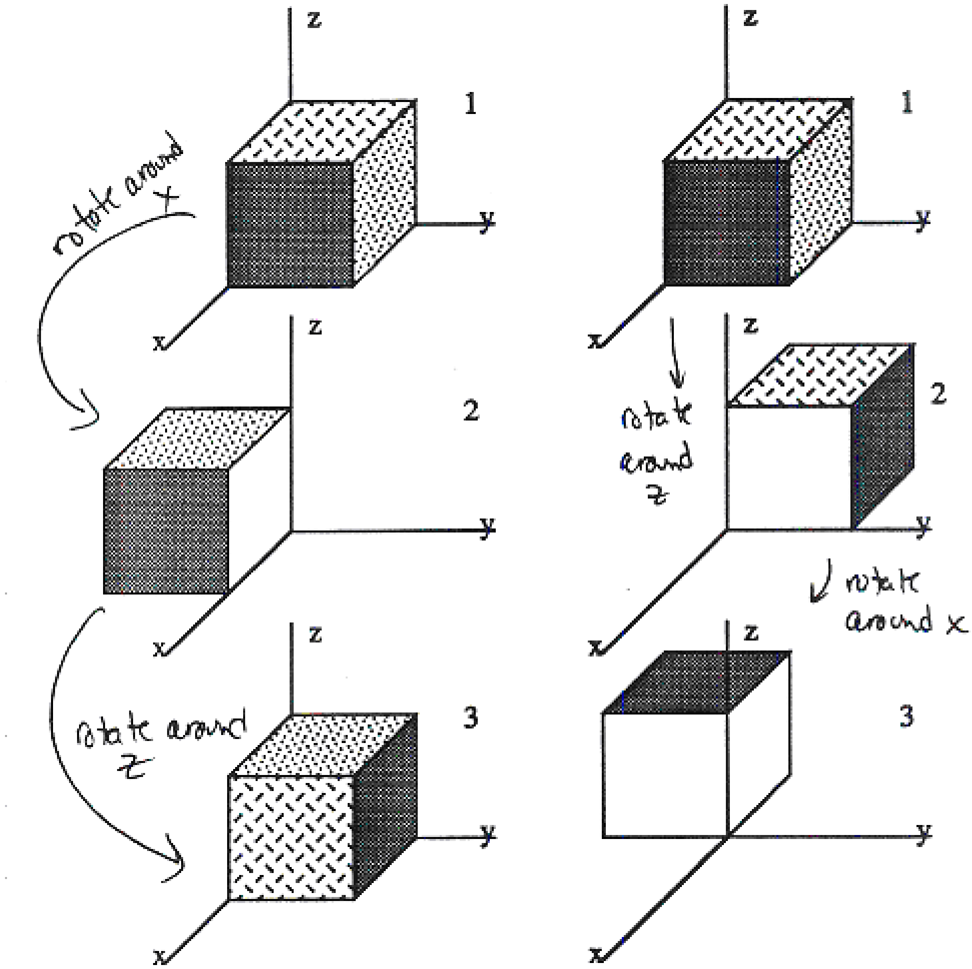
Successive rotations in three dimensions

Successive rotations (where the two rotation axes might be different) can be described like so:

$$\vec{a}'' = \underset{\sim}{R}_2\left[\underset{\sim}{R}_1\vec{a}\right].$$

In general, rotations do not commute:



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Kronecker delta symbol

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

The Kronecker delta symbol δ_{ij} gives us another way of writing dot products: $\vec{a} \cdot \vec{b} = \sum_{ij} a_i b_j \delta_{ij}$

Also, since

$$\hat{x}' \cdot \hat{x}' = 1, \quad \hat{y}' \cdot \hat{y}' = 1, \quad \hat{z}' \cdot \hat{z}' = 1$$
$$\hat{x}' \cdot \hat{y}' = 0, \quad \hat{y}' \cdot \hat{z}' = 0, \quad \hat{z}' \cdot \hat{x}' = 0$$

we have

$$\sum_{k=1}^{3} R_{ki} R_{kj} = \delta_{ij} \quad \text{and} \quad \sum_{k=1}^{3} R_{ik} R_{jk} = \delta_{ij} .$$

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Levi-Civita symbol

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } ijk \text{ is an even permutation of 123 } (123,\ 312,\text{ or } 231) \\ -1 & \text{if } ijk \text{ is an odd permutation of 123 } (213,\ 132,\text{ or } 321) \\ 0 & \text{if any two (or more) of } i, j, k \text{ are equal} \end{cases}$$

It's not hard to see that

$$\vec{a} \times \vec{b} = \sum_{i=1}^{i=3} \sum_{j=1}^{j=3} \sum_{k=1}^{k=3} \varepsilon_{ijk}\, a_i b_j\, \hat{k} \quad \left(\text{also written as } \sum_{i,j,k} \varepsilon_{ijk}\, a_i b_j\, \hat{k} \right).$$

When working with these symbols, you can always write things out explicitly, replacing the indices *i*, *j*, and *k* with numbers and writing out each term in the sums. It's messy, but clear.

One can even show that $\sum_k \varepsilon_{ijk}\varepsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$ should you be so moved!

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Tensors

Let's say we constructed a 3 × 3 object out of vectors like so: Start with $\vec{a}, \vec{b}$; define $\underset{\sim}{T}$ so that

$$\underset{\sim}{T} = \begin{bmatrix} a_x b_x & a_x b_y & a_x b_z \\ a_y b_x & a_y b_y & a_y b_z \\ a_z b_x & a_z b_y & a_z b_z \end{bmatrix}$$

In terms of components, $T_{ij} = a_i b_j$. (This is an "outer product.") If we switched to a (rotated) coordinate system, determined the coordinates of $\vec{a}', \vec{b}'$ in this system, we'd calculate $\underset{\sim}{T}'$ using the rule $T'_{ij} = a'_i b'_j$.

How is $\underset{\sim}{T}'$ related to $\underset{\sim}{T}$?

We know: $\vec{a}' = \underset{\sim}{R}\vec{a}$ and $\vec{b}' = \underset{\sim}{R}\vec{b}$.

In terms of components, $a'_i = \sum_k R_{ik} a_k$ and $b'_j = \sum_l R_{jl} b_l$

This comes from the definition of matrix multiplication.

As a result,

$$T'_{ij} = a_i' b_j' = \left(\sum_k R_{ik} a_k \right) \left(\sum_l R_{jl} b_l \right).$$

For a particular choice of $i, j, k, l,$ each of R_{ik}, R_{il}, a_k, b_l are just numbers so we can change the order of our sums:

$$T'_{ij} = a_i' b_j' = \left(\sum_k R_{ik} a_k \right) \left(\sum_l R_{jl} b_l \right) = \left\{ \sum_k R_{ik} a_k \left(\sum_l R_{jl} b_l \right) \right\}$$

$$= \sum_l \left[\sum_k \left(R_{ik} a_k R_{jl} b_l \right) \right] = \sum_k \left[\sum_l \left(R_{ik} R_{jl} a_k b_l \right) \right].$$

If we want, we can group things:

$$T'_{ij} = \sum_l \left[\underbrace{\left(\sum_k R_{ik} T_{kl} \right)} R_{jl} \right]$$

This is the il^{th} component of the product $\underset{\sim}{R} \times \underset{\sim}{T}$.

The *transpose* of $\underset{\sim}{R}$ is defined this way: $R^T_{ij} = R_{ji}$. (Useful: the transpose of a rotation matrix is also its inverse.)

This lets us write

$$T'_{ij} = \sum_l \left[\left(\underset{\sim}{R} \underset{\sim}{T} \right)_{il} \left(R^T \right)_{lj} \right].$$

From the definition of matrix multiplication, the sum is simply the ij^{th} component of the product of the matrices $\underset{\sim}{R}\underset{\sim}{T}$ and $\underset{\sim}{R}^T$.

As a result,

$$\underset{\sim}{T}' = \left\{\underset{\sim}{R}\ \underset{\sim}{T}\right\}\underset{\sim}{R}^T = \underset{\sim}{R}\ \underset{\sim}{T}\ \underset{\sim}{R}^T.$$

I find I prefer to write it in terms of components:

$$T'_{ij} = \sum_{kl} R_{ik} R_{jl} T_{kl}.$$

Something that transforms this way is called a tensor (of rank 2).

We'll see later how tensors can be useful.

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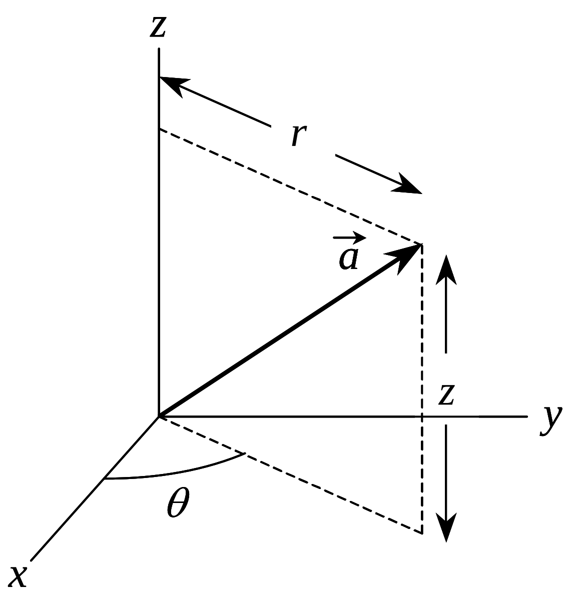
7. Non-cartesian coordinate systems

Sometimes these will be convenient: don't be put off by their unfamiliarity!

cylindrical coordinates



$$x = r\cos\theta$$

$$y = r\sin\theta$$

$r,\ \theta,\ z$

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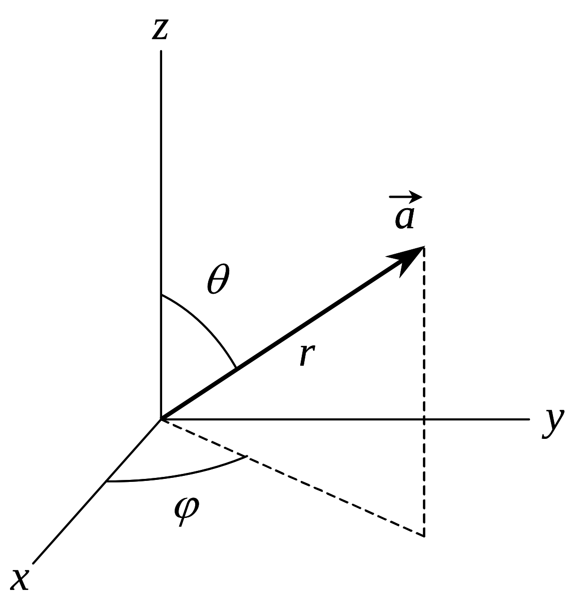
spherical coordinates



$$z\ =\ r\cos\theta$$

$$x\ =\ r\sin\theta\ \cos\varphi$$

$$y\ =\ r\sin\theta\ \sin\varphi$$

$r,\ \theta,\ \varphi$

The unit vectors changes which way they point as $\vec{x}$ moves around in space if we are working in a non-Cartesian coordinate system. This complicates the taking of derivatives.

Note the unfortunate change in the meaning of the angle θ when we switch from cylindrical to spherical coordinates.

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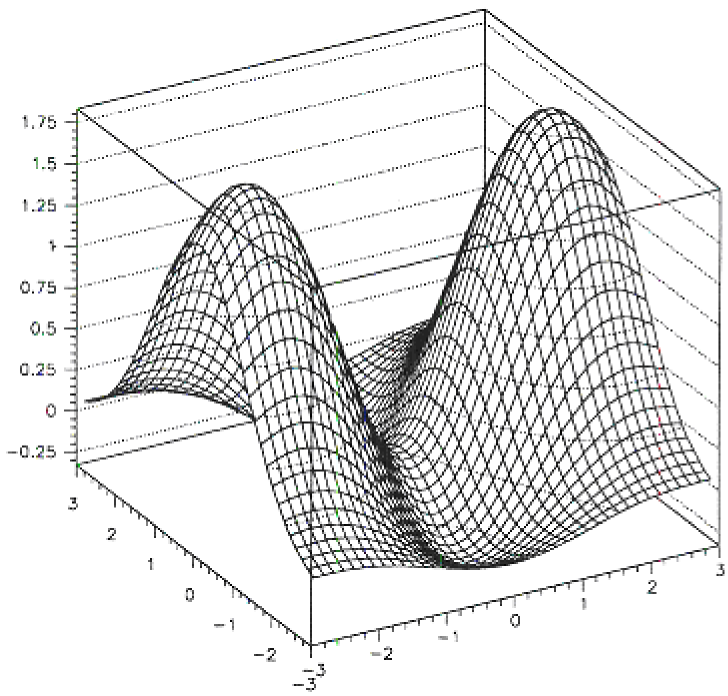
8. partial derivatives

We'll work with functions of several variables. Sometimes we'll want to know how the function changes if we change one variable while keeping all the others fixed.

For example, imagine we define a function *h*(*x*,*y*) that represents the height above sea level in Champaign as a function of latitude (*x*) and longitude (*y*).

Here's a graph of the function $h = \left(x^2 - 0.3y^2\right)e^{-\left(\frac{x^2}{5} - \frac{y^2}{3}\right)}$.



$$h = \left(x^2 - 0.3y^2\right)e^{-\left(\frac{x^2}{5} - \frac{y^2}{3}\right)}$$

The definition of a partial derivative with respect to *y* is this:

$$\frac{\partial h(x,y)}{\partial y} \equiv \lim_{\Delta y \to 0} \frac{h(x, y+\Delta y) - h(x,y)}{\Delta y}$$

Note that <u>*x* is held constant</u>.

This amounts to measuring the slope as (in this case) you move parallel to the y axis.

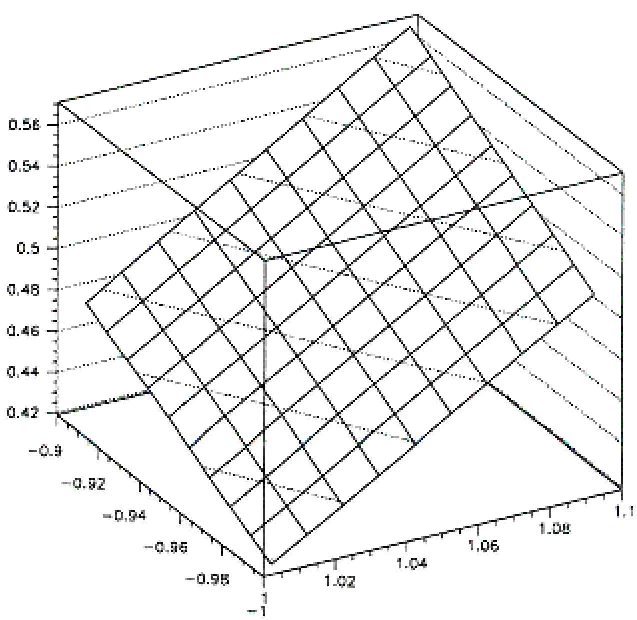
It's easy to take partials: you treat all the variables except the selected one as if they were constants.

For example, if $f(x,y,z) = x^2 + 6xy^3z + xz^2$ then

$$\frac{\partial f}{\partial y} = \frac{\partial\left(x^2\right)}{\partial y} + \frac{\partial 6xy^3z}{\partial y} + \frac{\partial xz^2}{\partial y}$$

$$= 0 + 18xy^2z + 0$$

If you look in a <u>small</u> region around a particular *x*,*y* point any smooth function will look like a plane:



$$h = \left(x^2 - 0.3y^2\right)e^{-\left(\frac{x^2}{5} - \frac{y^2}{3}\right)}$$

restricted to the region
$1 \le x \le 1{\cdot}1,\ -1 \le y \le 0.9$

———— ✳ ————

using partial derivatives to approximate changes in functions

How much does *h* change if we go from x,y to $x+\Delta x,\ y+\Delta y$?

$$\Delta h = h\left(x+\Delta x, y+\Delta y\right) - h\left(x,y\right)$$
$$= \left\{h\left(x+\Delta x,\ y+\Delta y\right) - h\left(x,\ y+\Delta y\right)\right\} + \left\{h\left(x,\ y+\Delta y\right) - h\left(x,y\right)\right\}$$

I've just subtracted, then added the same thing.

For small Δx, Δy we have

$$h(x+\Delta x,\ y+\Delta y)-h\left(x,\ y+\Delta y\right)\approx\frac{\partial h\ (x,y+\Delta y)}{\partial x}\ \Delta x$$

from the definition of a partial derivative.

Also, as long as the partial derivative doesn't change violently with position,

$$\frac{\partial h(x,y+\Delta y)}{\partial x}\ \approx\ \frac{\partial h(x,y)}{\partial x}.$$

As a result, $\Delta h\ \approx\ \frac{\partial h}{\partial x}\Delta x\ +\ \frac{\partial h}{\partial y}\Delta y$.

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gradient operator

The <u>gradient operator</u> (in two dimensions) is defined this way:

$$\vec{\nabla}\ \equiv\ \hat{x}\frac{\partial}{\partial x}\ +\ \hat{y}\frac{\partial}{\partial y}$$

We use it like so:

$$\vec{\nabla} h(x,y) = \hat{x}\frac{\partial h}{\partial x}(x,y) + \hat{y}\frac{\partial}{\partial y}(x,y)$$

If we move a small distance away from the point (x,y) along the direction $\vec{\delta} = \hat{x}\delta_x + \hat{y}\delta_y$ we find…

$$\Delta h = \frac{\partial h}{\partial x}\delta_x + \frac{\partial h}{\partial y}\delta_y \; .$$

With our definition for $\vec{\nabla}$, we can rewrite this as

$$\Delta h = \vec{\delta}\cdot\left(\vec{\nabla} h\right) .$$

In three (Cartesian) dimensions

$$\vec{\nabla} f\left(x,y,z\right) \equiv \left[\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right] f\left(x,y,z\right) .$$

Useful fact: the <u>direction</u> of $\vec{\nabla} h$ is the direction in which $h(x,y)$ changes most rapidly.

In cylindrical coordinates $r,\ \theta,\ z$ we have

$$\vec{\nabla} f\left(r,\theta,z\right) \equiv \left[\hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial \theta} + \hat{z}\frac{\partial}{\partial z}\right] f\left(r,\theta,z\right).$$

In spherical coordinates $r,\ \theta,\ \varphi$ we have

$$\vec{\nabla} f(r,\theta,\varphi) \equiv \left[\hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial \theta} + \frac{\hat{\varphi}}{r\sin\theta}\frac{\partial}{\partial \varphi} \right] f(r,\theta,\varphi).$$

——————— ✳ ———————

divergence operator

The <u>divergence operator</u> acts on vector fields and is defined as specified below.

Cartesian coordinates:

$$\vec{A}(x,y,z) \equiv A_x(x,y,z)\hat{x} + A_y(x,y,z)\hat{y} + A_z(x,y,z)\hat{z}$$

$$\vec{\nabla}\cdot\vec{A}(x,y,z) \equiv \frac{\partial A_x(x,y,z)}{\partial x} + \frac{\partial A_y(x,y,z)}{\partial y} + \frac{\partial A_z(x,y,z)}{\partial z}$$

Cylindrical coordinates:

$$\vec{A}(r,\theta,z) \equiv A_r(r,\theta,z)\hat{r} + A_\theta(r,\theta,z)\hat{\theta} + A_z(r,\theta,z)\hat{z}$$

$$\vec{\nabla}\cdot\vec{A}(r,\theta,z) \equiv \frac{1}{r}\frac{\partial(rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}$$

Spherical coordinates:

$$\vec{A}(r,\theta,\varphi) \equiv A_r(r,\theta,\varphi)\hat{r} + A_\theta(r,\theta,\varphi)\hat{\theta} + A_z(r,\theta,\varphi)\hat{\varphi}$$

$$\vec{\nabla}\cdot\vec{A}(r,\theta,\varphi) \equiv \frac{1}{r^2}\frac{\partial(r^2A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(A_\theta\sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial \varphi}$$

curl operator

The curl operator acts on vector fields. Its form in Cartesian coordinates is this:

$$\vec{\nabla}\times\vec{A}(x,y,z) \equiv \left[\frac{\partial A_y(x,y,z)}{\partial x}-\frac{\partial A_x(x,y,z)}{\partial y}\right]\hat{z} + \left[\frac{\partial A_z(x,y,z)}{\partial y}-\frac{\partial A_y(x,y,z)}{\partial z}\right]\hat{x} + \left[\frac{\partial A_x(x,y,z)}{\partial z}-\frac{\partial A_z(x,y,z)}{\partial x}\right]\hat{y}$$

The forms in cylindrical and spherical coordinates are more complicated in appearance. See a math reference for them.

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9. quadratic formula

Very handy: if $ax^2+bx+c=0$ then $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$.

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