Physics 199 Physics of Music

Lecture Notes Week II

[Chapter II — Simple Vibrating Systems]

In order to produce a sound, an object (e.g. a musical instrument) must be made to vibrate, by whatever means possible. This vibration is (clearly) mechanical in nature.

The vibrating object couples to the air surrounding it, sound waves in the air are created, which propagate radially outwards from the source (the vibrating object) to an observer's ear(s). Thus a sound is heard (perceived) by the observer.

A simple example of a vibrating system is a mass on a spring (a crude model of a vibrating musical instrument – the so-called "spherical cow approximation") which undergoes so-called 1-D simple harmonic motion:

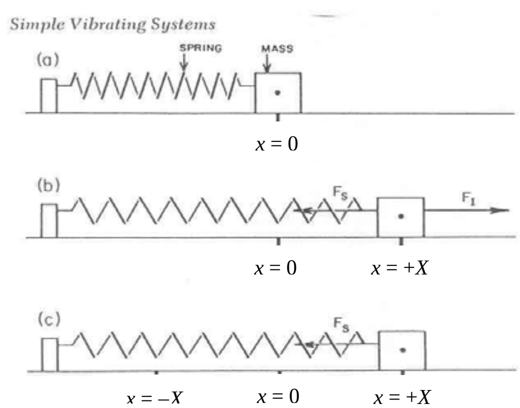


FIG. 1. A mass attached to a spring and resting on a smooth table. (a) Equilibrium position. (b) Mass displaced and held in new position. (c) Mass released.

If the mass M is horizontally displaced from its equilibrium (x = 0) position by pulling on it to the right, as shown in the above figure. The force necessary to accomplish this is $\boxed{F_I = +kX}$, where k = the so-called "spring constant" of the spring (k has metric units of Newtons/meter) and X = the initial displacement of the mass M from its x = 0 equilibrium position.

At time t = 0 the mass is released. At that instant, the only {horizontal} force acting on the mass is due to the restoring force of the spring: $\boxed{F_S(t=0) = -kX = -F_I}$. However, from Newton's 2nd Law $F = Ma$, therefore at time t = 0: $\boxed{F_S(t=0) = -kX = Ma(t=0)}$.

As time progresses the mass M oscillates horizontally back and forth about its x = 0 equilibrium position, exhibiting sinusoidal/harmonic motion. Mathematically, the time-dependence of this horizontal sinusoidal/harmonic motion is described by:

Longitudinal displacement from equilibrium: $\boxed{x(t) = X\cos(2\pi ft) = X\cos(\omega t)}$ (m)

(meters) displacement amplitude (meters) frequency of oscillation (cycles per second = Hertz) cps Hz

Omega: $\omega \equiv 2\pi f$ = so-called angular frequency (units = radians per second)

Period of oscillation: $\tau \equiv \frac{1}{f} = \frac{2\pi}{\omega}$ (seconds)

The instantaneous horizontal speed of the moving mass *v*(*t*) with time *t* is defined as the time rate of change of the horizontal position (longitudinal displacement) of the moving mass with time *t*, physically, *v*(*t*) is the instantaneous local slope of the *x*(*t*) vs. *t* graph at time *t*:

$$v(t) = \frac{\Delta x(t)}{\Delta t} = \frac{dx(t)}{dt} = \text{so-called derivative of } x \text{ with respect to time, } t.$$

$$v(t) = \frac{d}{dt}(x(t)) = \frac{d}{dt}\left[X\cos(2\pi ft)\right] = -2\pi fX\sin(2\pi ft) = -\omega X\sin(\omega t) = -V\sin(\omega t)$$

We see that: $V = \omega X = 2\pi fX$

i.e. the speed "amplitude", *V* = max speed is related to the displacement amplitude, *X* by this formula

Instantaneous Horizontal Speed of the Moving Mass: $v(t) = -V\sin(2\pi ft) = -V\sin(\omega t)$ (m/s)

$v(t)$: (meters/sec); V: speed amplitude (m/s); f: frequency of oscillation (cycles per second = Hertz)

The instantaneous horizontal acceleration of the moving mass *a*(*t*) with time *t* is defined as the time rate of change of the horizontal speed of the moving mass with time *t*, physically, *a*(*t*) is the instantaneous local slope of the *v*(*t*) vs. *t* graph at time *t*:

$$a(t) = \frac{\Delta v(t)}{\Delta t} = \frac{dv(t)}{dt} = \text{so-called derivative of } v \text{ with respect to time, } t.$$

$$a(t) = \frac{d}{dt}(v(t)) = \frac{d}{dt}\left[-V\sin(2\pi ft)\right] = -2\pi fV\cos(2\pi ft) = -\omega V\cos(\omega t) = -A\cos(\omega t)$$

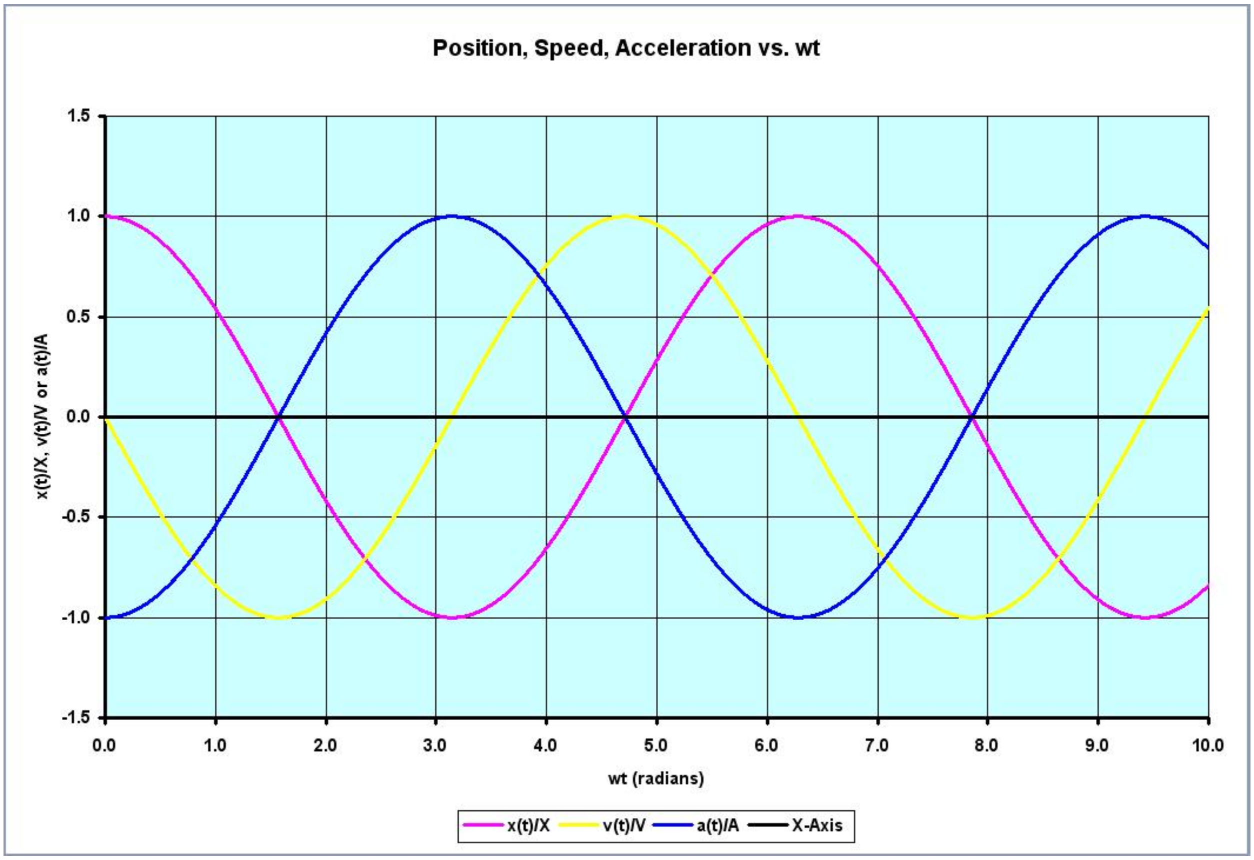
We see that: $A = \omega V = 2\pi fV$ but: $V = \omega X = 2\pi fX$ $\therefore$ $A = \omega^2 X = (2\pi f)^2 X$

i.e. the acceleration "amplitude", *A* = max acceleration is related to the displacement amplitude, *X* by this formula

Instantaneous Horizontal Accel. of the Moving Mass: $a(t) = -A\cos(2\pi ft) = -A\cos(\omega t)$ (m/s^2)

$a(t)$: (meters/sec^2); A: acceleration amplitude (m/s^2); f: frequency of oscillation (cycles per second = Hertz)

The time dependence of the longitudinal position, *x(t)* (i.e. displacement of the mass from its equilibrium position) vs. time, *t* and longitudinal speed of the mass *v(t)* vs. time, *t* is shown in the figure below; note that each has been normalized to their respective amplitudes:



Once the mass M has been set in motion, Newton's 2nd Law tells us: $F(t) = -kx(t) = Ma(t)$

But: $x(t) = X\cos(2\pi ft) = X\cos(\omega t)$ and: $a(t) = -A\cos(2\pi ft) = -A\cos(\omega t)$

However, from above, we also know that: $A = \omega^2 X = (2\pi f)^2 X$ $\therefore$ $-kX = -\omega^2 MX$

Thus, the frequency f and angular frequency ω of oscillation of the mass M on the spring are:

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{M}}$$ Cycles per second, or Hz and $$\omega = 2\pi f = \sqrt{\frac{k}{M}}$$ (radians/sec)

The period of oscillation τ of the mass M on the spring is: $$\tau = \frac{1}{f} = 2\pi\sqrt{\frac{M}{k}}$$ (seconds)

The instantaneous <u>potential</u> energy *<u>stored</u>* in the stretched/compressed spring is:

$$P.E.(t) = \frac{1}{2}kx^2(t)$$ (Joules)

The instantaneous <u>kinetic</u> energy associated with the *<u>moving</u>* mass, M is:

$$K.E.(t) = \frac{1}{2}Mv^2(t)$$ (Joules)

The potential energy of the spring and the kinetic energy of the moving mass are both time dependent:

$$P.E.(t) = \frac{1}{2}kx^2(t) = \frac{1}{2}kA^2\cos^2(\omega t)$$
$$K.E.(t) = \frac{1}{2}Mv^2(t) = \frac{1}{2}MV^2\sin^2(\omega t)$$

<u>However</u>: $V = \omega X$ and: $\omega = \sqrt{\frac{k}{M}}$ or: $k = M\omega^2$

<u>Thus</u>:

$$P.E.(t) = \frac{1}{2}kx^2(t) = \frac{1}{2}kX^2\cos^2(\omega t)$$
$$K.E.(t) = \frac{1}{2}Mv^2(t) = \frac{1}{2}MV^2\sin^2(\omega t) = \frac{1}{2}M\omega^2X^2\sin^2(\omega t) = \frac{1}{2}kX^2\sin^2(\omega t)$$

Let us define: $$E_o \equiv \frac{1}{2}kX^2 = \frac{1}{2}M\omega^2X^2$$

<u>Then</u>:
$$P.E.(t) = E_o\cos^2(\omega t)$$
$$K.E.(t) = E_o\sin^2(\omega t)$$

We define the *<u>total</u>* energy, E_{Tot} as:

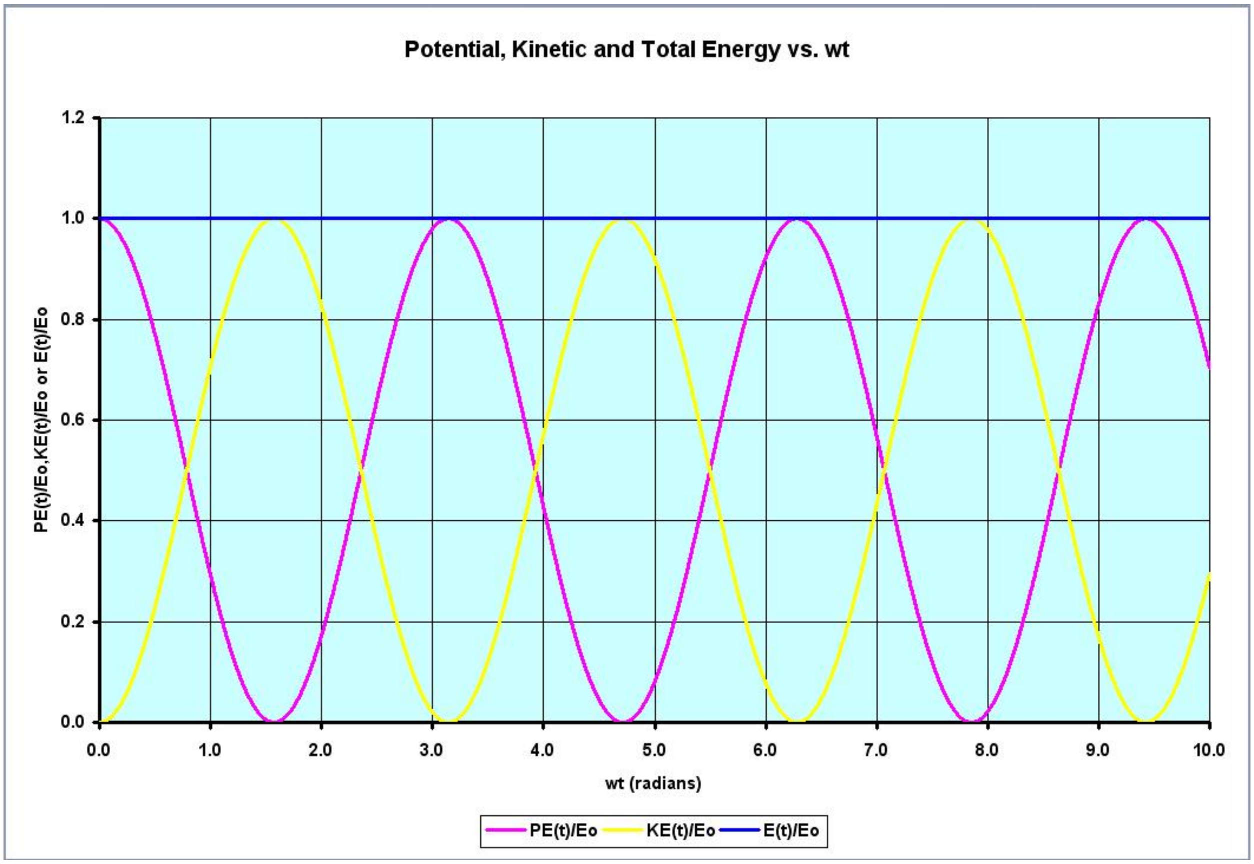
$$E_{Tot}(t) = P.E.(t) + K.E.(t) = E_o\cos^2(\omega t) + E_o\sin^2(\omega t) = E_o\left\{\cos^2(\omega t) + \sin^2(\omega t)\right\}$$

Using the trigonometric identity $1 = \cos^2 x + \sin^2 x$ we see that:

$$\boxed{E_{Tot}(t) = E_o = \frac{1}{2}kX^2 = \frac{1}{2}M\omega^2 X^2} = \text{constant!}$$

The total energy in (spring + mass) system *is* constant — due to conservation of energy!!

Graphs of *P.E.*(t), *K.E.*(t), and $E_{Tot}(t)$ vs. time (all normalized to E_o):



Note that the *P.E.*(t), *K.E.*(t), and $E_{Tot}(t)$ are all always > 0 (i.e. never negative)!!!

A real vibrating spring — mass system suffers from various energy loss mechanisms:

- friction – mass slides on surface
  – mass slides through viscous air
- spring also dissipates energy internally each time it is flexed

— Motion of mass on a spring is damped by friction

— Original energy, $E_{Tot} = E_o$ is dissipated by friction, internal losses in spring (another type of friction)

— Initial energy E_o ultimately winds up as heat (another form of energy) thus the mass, spring, surface and air all heat up with time…

Damping by friction affects the vibrational motion:

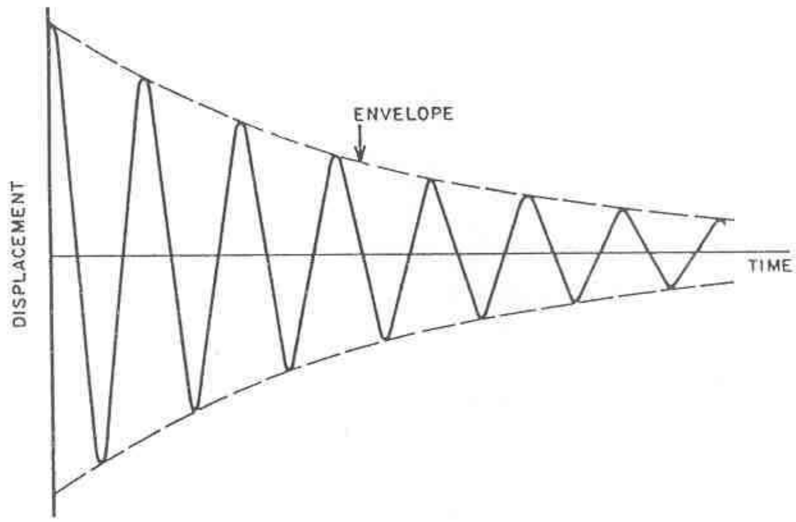


FIG. 3. Graph of displacement versus time for a damped vibration.

Damping processes (e.g. due to friction) usually lower the frequency of oscillation of a vibrating system. Small damping – slight decrease in the oscillation frequency. Very heavy damping – no oscillation(s) at all!

More realistic motion of a vibrating mass on spring — e.g. driving it with periodic force

- have to get mass moving first (initially at rest)
- takes a while for oscillations to build up
- reaches a <u>steady state</u> displacement amplitude
- switch off the driving force, displacement amplitude decays away

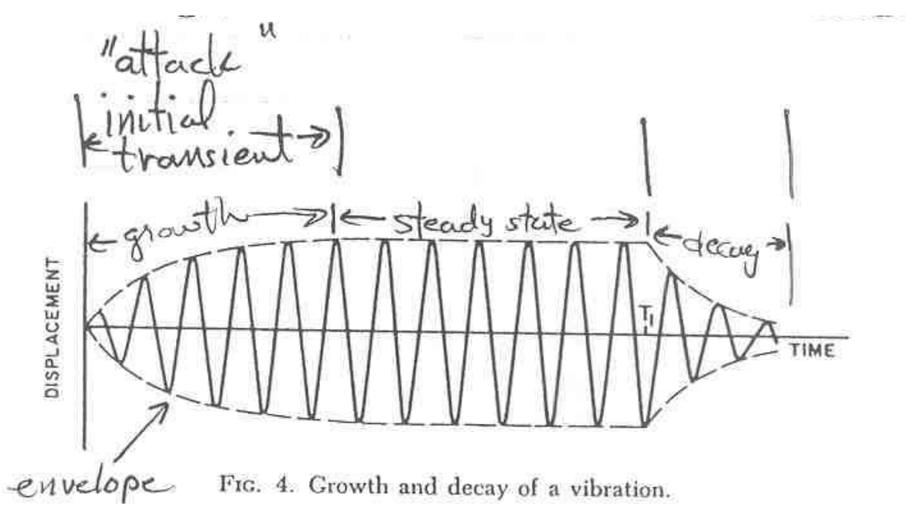


FIG. 4. Growth and decay of a vibration.

Vibrations of a tuning fork. Tongs both have mass *M* and length *L*. Resonance frequency depends on *M*, *L* and the stiffness of the material making up the tuning fork.

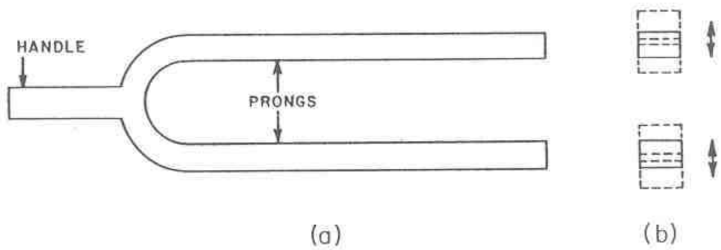


FIG. 5. (a) Tuning fork. (b) End view of vibrating fork.

Can graphically map out vibrations of tuning fork using the following method:

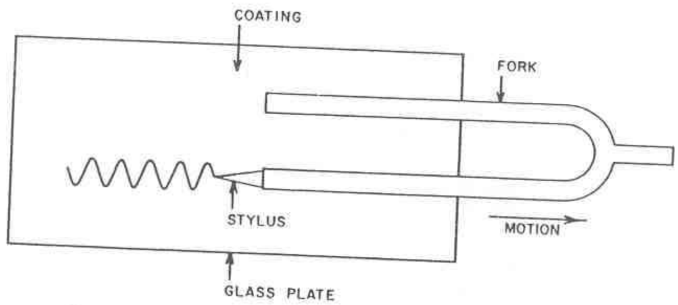


FIG. 6. A tuning fork drawing a graph of its own vibration.

NOTE:
Figures are taken from the course text "*The Acoustical Foundations of Music*" by John Backus, second Edition.

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