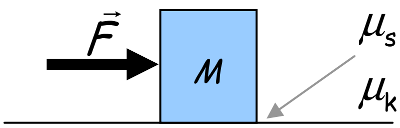
# Friction and Uniform Circular Motion

## The Plan

Today we continue to practice Newton's Laws, but this time we have introduced a new force… friction. Static friction is a constraint force that is just enough to keep an object from sliding. Kinetic friction has a predictable magnitude that fights the relative sliding of two surfaces. Friction forces will appear in free body diagrams this week and will contribute to any other forces pushing on an object. The key to remember is that Newton's Second Law, $\vec{F}_{net} = M\vec{a}$ , is still the key to analyzing the motion of objects. We just have some new terms in $F_{net}$.

We have also introduced uniform circular motion this week. If you know an object is in uniform circular motion, you know its acceleration is $v^2/r$ toward the center of the circle… and that's all. There must be some forces on the object pulling or pushing it to the center of that circle. We can relate the forces on the object to this centripetal acceleration $v^2/r$ by… you guessed it… Newton's Second Law. We can just plug in $v^2/r$ for the acceleration toward the center and proceed like any other problem with a known acceleration.

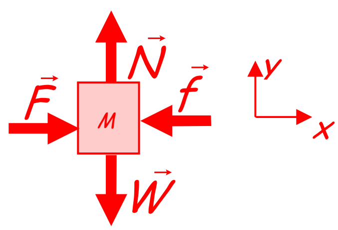
The problems this week gradually build up your understanding and practice with frictional forces. The first problem starts with a single object with different applied forces. After this basic practice, we revisit the preflight problem of two stacked blocks to find the magnitude and direction of frictional forces using Newton's Laws. If your group has time, we end with a uniform circular motion problem before taking the quiz over Newton's Laws and frictional forces.




DQ1) A block of mass $M$=1 kg rests on a table with coefficient of static friction $\mu_s = 0.5$ and coefficient of kinetic friction $\mu_k = 0.25$. A force $\vec{F}$ is applied to the right. Work with your group to find the magnitude and direction of the frictional force $f$ and acceleration $a$ for each of the following values of the applied force's magnitude.

The maximum force that static friction can provide is given by $f_{s,\,max} = \mu_s N$. We need to find the normal force on the box to determine the maximum force that static friction can provide. If the applied force in the x-direction is less than this maximum, then we know that the mass will not slip (in this problem, that means the block's acceleration will be zero). If the applied force is larger than the maximum that static friction can provide, then the mass will slide, an we must then use kinetic friction.

To find the maximum force that static friction can provide, we begin with a free body diagram for the mass.




We choose the axes to be horizontal and vertical because we know that if the block does accelerate, it will be along the surface in the horizontal direction, along our x-axis. In addition, this choice makes all of our forces parallel to one of the axes.

To find the normal force, we'll use Newton's Second Law in the vertical direction. We know that the vertical acceleration of the block is zero.

$$\vec{F}_{net} = M\vec{a}$$

$$F_{net,y} = Ma_y$$

$$F_y + N_y + W_y + f_y = M(0)$$

$$0 + N + (-Mg) + 0 = 0$$

$$N = Mg$$

The maximum force that static friction can provide is therefore:

$$f_{s,max} = \mu_s N = \mu_s (Mg) = (0.5)(1\ \text{kg})\left(9.81\tfrac{\text{m}}{\text{s}^2}\right) = 4.9\ \text{N}$$

If the x-component of all the other forces added together is less than this maximum, the block will not slide.

a) $F = 1\text{ N}$

1 N < 4.9 N (max static friction)

Since 1 Newton is smaller than the maximum force that static friction can provide, the block does not slide. Its acceleration is therefore zero. a=0

$f_{\text{Friction}} = 1\text{ N leftward}$

$a = 0\,\frac{\text{m}}{\text{s}^2}$

Now that we know the acceleration, we can use the free body diagram above with Newton's Second Law to determine the static frictional force on the block.

$$F_{\text{net},x} = Ma_x$$

$$F_x + N_x + W_x + f_{\text{static},x} = M(0)$$

$$F + 0 + 0 + (-f_{\text{static}}) = 0$$

$$f_{\text{static}} = F = 1\text{ N}$$

b) $F = 4\text{ N}$

4 N < 4.9 N (max static friction)

Since 4 Newtons is smaller than the maximum force that static friction can provide, the block does not slide. Its acceleration is therefore zero. a=0

$f_{\text{Friction}} = 4\text{ N leftward}$

$a = 0\,\frac{\text{m}}{\text{s}^2}$

Now that we know the acceleration, we can use the free body diagram above with Newton's Second Law to determine the static frictional force on the block.

$$F_{\text{net},x} = Ma_x$$

$$F_x + N_x + W_x + f_{\text{static},x} = M(0)$$

$$F + 0 + 0 + (-f_{\text{static}}) = 0$$

$$f_{\text{static}} = F = 4\text{ N}$$

c) $F = 7.5\ \text{N}$

7.5 N > 4.9 N

Here, the applied force exceeds the maximum that the static frictional force can apply. The block will therefore slide across the floor, and we must use *kinetic* friction.

$$f_{\text{Friction}} = 2.45\ \text{N leftward}$$

$$a = 5.05\ \tfrac{\text{m}}{\text{s}^2}$$

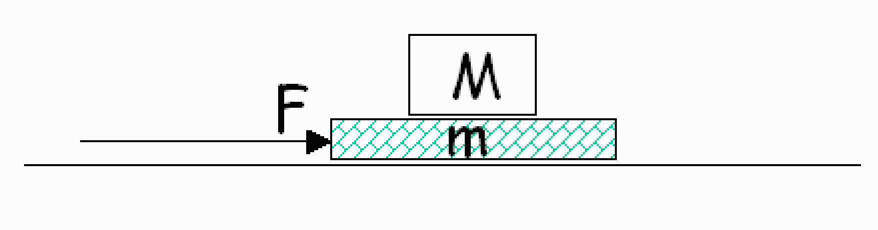
With kinetic friction, we know what the magnitude of the force is (assuming we know the normal force).

$$\begin{aligned} f_{\text{kinetic}} &= \mu_k N = \mu_k (Mg) \\ &= (0.25)(1\ \text{kg})\left(9.81\tfrac{\text{m}}{\text{s}^2}\right) \\ &= 2.45\ \text{N} \end{aligned}$$

Since we know all the forces in the x-direction, we can calculate what the acceleration of the mass is.

$$\begin{aligned} F_{\text{net},x} &= Ma_x \\ F_x + N_x + W_x + f_{\text{kinetic},x} &= M(a) \\ F + 0 + 0 + (-\mu_k N) &= Ma \\ F - \mu_k (Mg) &= Ma \\ a &= \frac{F}{M} - \mu_k g \\ &= \frac{7.5\ \text{N}}{1\ \text{kg}} - (0.25)\left(9.81\tfrac{\text{m}}{\text{s}^2}\right) \\ &= 5.05\tfrac{\text{m}}{\text{s}^2} \end{aligned}$$

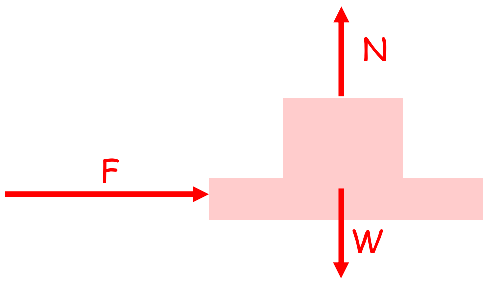
DQ2) A block of mass *M* rests on another block of mass *m* ( *M* > *m*) as shown below. A constant force F is applied to the bottom block (mass *m*) and both blocks are observed to move together with constant acceleration *a*. Let *f* represent the frictional force the bottom block (*m*) exerts on the top block (*M*). The table is frictionless.




a) Determine the acceleration of the two blocks in terms of *F, M, m* and *g*. Check your answer with the rest of your group.

We'll start by drawing a free body diagram for the two-block system.

a = $\dfrac{F}{m+M}$




We can find the acceleration of this object in the x-direction using Newton's Second Law:

$$F_{\text{net on total},x} = M_{\text{total}} a_x$$

$$F_x + N_x + W_x = (m+M)a_x$$

$$F + 0 + 0 = (m+M)a_x$$

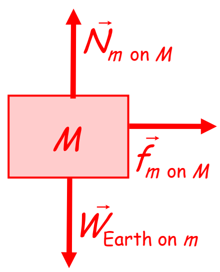
$$a_x = \frac{F}{m+M}$$

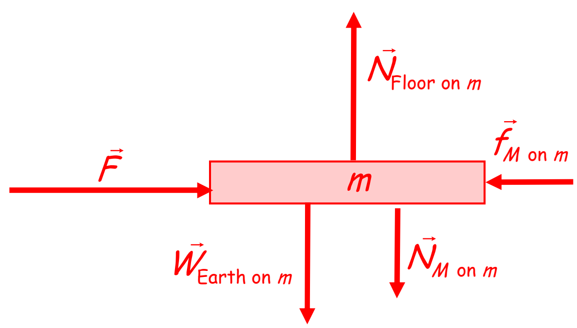
b) Working with your group, draw the free body diagrams for each of the masses. Check each others' diagrams with Newton's 2nd Law. Can the forces you drew explain the acceleration of each object?

To find the direction of the frictional force on the top mass *M*, we used its known acceleration (to the right) and Newton's Second Law.

$F_{\text{net on }M} = Ma$

$f_{m \text{ on } M} = Ma$

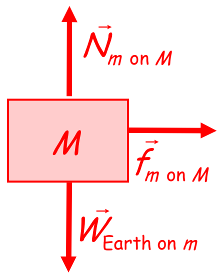



To accelerate to the right, the net force on the top mass must point to the right. Friction from *m* is the only force on the top block in the horizontal direction; the frictional force must therefore point to the right to make *M* accelerate to the right.

c) Find an expression for $f_{m \text{ on } M}$, the magnitude of the frictional force that $m$ exerts on $M$ in terms of the variables $F$, $M$, $m$ and $g$. Also state the direction of this frictional force.

We'll start with the free body diagram of the top block.




$$f_{m \text{ on } M} = M\left(\frac{F}{m+M}\right)$$

direction = right

The only force that has an x-component is the frictional force on the top block from the bottom one. Since we already know the acceleration of the blocks, we can use Newton's Second Law on the top block in the x-direction to find the magnitude of the frictional force.

$$F_{net,x} = Ma_x$$

$$N_{m \text{ on } M,x} + W_{\text{Earth on } M,x} + f_{m \text{ on } M,x} = M\left(\frac{F}{m+M}\right)$$

$$0 + 0 + f_{m \text{ on } M} = M\left(\frac{F}{m+M}\right)$$

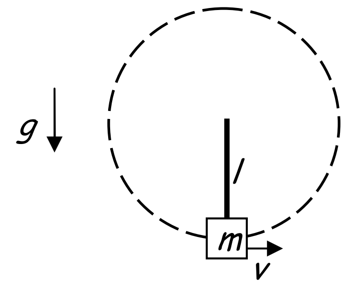
$$f_{m \text{ on } M} = M\left(\frac{F}{m+M}\right)$$

d) Using your intuition, if we stopped pushing the block then what would happen to the force $f_{m \text{ on } M}$? Does your equation correctly predict this outcome if you plug in $F=0$?

If we stopped pushing on the blocks, they would no longer be accelerating to the right. They would both travel to the right at constant velocity. Since they would both have the same velocity, neither would try to slip relative to the other… no friction would be necessary to keep them moving at the same velocity. We would therefore expect the frictional force to go to zero if the applied force were zero.

We can check our expression by sending the applied force to zero and seeing what we get for the frictional force on the top block from the bottom one.

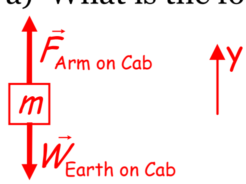
$$f_{m \text{ on } M} = \frac{M}{m+M}F \rightarrow \frac{M}{m+M}0 = 0 \checkmark$$




DQ3) A ride at a carnival has a cab of mass $m$ attached to a metal arm (length $l$) that rotates in a vertical circle at constant speed $v$. Talk with your group to find the force from the arm on the cab ($F$) at the indicated positions. Give your answers in terms of $m$, $l$, $v$, and $g$.

Since the cab travels in uniform circular motion, we know that its acceleration must have the magnitude $v^2/r$ and be pointed to the center of the circle. The arm must provide enough force to make this happen.

a) What is the force from the arm on the cab when the cab is at its lowest point?




We've drawn the free body diagram for the cab at the bottom of its swing. We choose up as our positive y-axis because this is the direction that the cab accelerates (toward the center of the circle). We'll use Newton's Second Law to relate the forces on the cab and its acceleration.

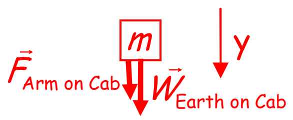
$$F_{\text{net},y} = ma_y$$

$$F_{\text{Arm on Cab},y} + W_{\text{Earth on Cab},x} = m\left(\frac{v^2}{l}\right)$$

$$F + (-mg) = m\frac{v^2}{l}$$

$$F = \boxed{m\frac{v^2}{l} + mg}$$

b) What is the force from the arm on the cab when the cab is at its highest point?




At this position, both the force from the arm and the weight point downward, toward the center of the circle. Since the cab accelerates downward, we choose down as the positive y-direction. Again, we'll use Newton's Second Law to relate the forces on the cab to its acceleration.

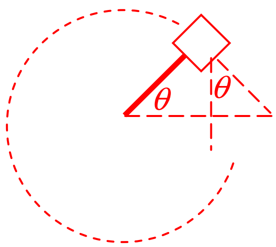
$$F_{\text{net},y} = ma_y$$

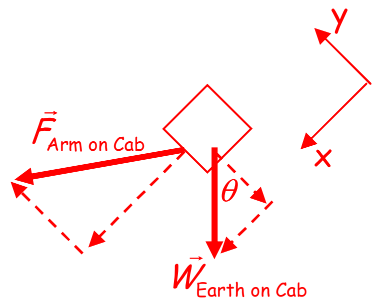
$$F_{\text{Arm on Cab},y} + W_{\text{Earth on Cab},x} = m\left(\frac{v^2}{l}\right)$$

$$F + (+mg) = m\frac{v^2}{l}$$

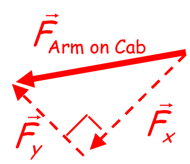
$$F = \boxed{m\frac{v^2}{l} - mg}$$

c) What is the force from the arm on the cab when the arm makes an angle of $\theta$ with the horizontal? ($\theta$ may appear in your answer).




Here we again choose toward the center of the circle as the positive axis because we know that's the direction the cab's acceleration must point (uniform circular motion). The force from the cab must have two components. One is an x-component toward the center of the circle to generate the centripetal acceleration. In addition, it must have a y-component to counteract the y-component of the weight. This latter component ensures that the y-acceleration is zero, keeping the tangential speed constant. We'll again use Newton's Second Law to relate the acceleration in each direction to the components of the force from the arm on the cab.

| | |
|---|---|
| Apply Newton's Second Law in the x-direction. We know that the acceleration of the cab in the x-direction is $v^2/l$ toward the center of the circle (uniform circular motion). | $F_{net,x} = ma_x$<br>$F_x + W_x = m\left(\frac{v^2}{l}\right)$<br>$F_x + (mg\sin\theta) = m\frac{v^2}{l}$<br>$F_x = m\frac{v^2}{l} - mg\sin\theta$ |
| Now apply Newton's Second Law in the y-direction. We know that the acceleration of the cab in the y-direction is zero (tangential speed never changes). | $F_{net,y} = ma_y$<br>$F_y + W_y = m(0)$<br>$F_y + (-mg\cos\theta) = 0$<br>$F_y = mg\cos\theta$ |
| Now we can combine the x- and y-components of the force from the arm on the cab to find the magnitude of the force form the arm on the cab. |  <br>$F^2 = F_x^{\ 2} + F_y^{\ 2}$<br>$F = \sqrt{F_x^{\ 2} + F_y^{\ 2}}$<br>$= \boxed{\sqrt{\left(m\frac{v^2}{l} - mg\sin\theta\right)^2 + (mg\cos\theta)^2}}$ |

| | |
|---|---|
| We can simplify this answer a little. | $F = \sqrt{\left(m\frac{v^2}{l}\right)^2 + \left(-mg\sin\theta\right)^2 - 2\left(m\frac{v^2}{l}\right)\left(mg\sin\theta\right) + \left(mg\cos\theta\right)^2}$ |
| Factor $(mg)^2$ out of two terms. | $= \sqrt{\left(m\frac{v^2}{l}\right)^2 - 2\left(m\frac{v^2}{l}\right)\left(mg\sin\theta\right) + \left(mg\right)^2\left(\cos^2\theta + \sin^2\theta\right)}$ |
| Factor $m^2$ out of the whole expression, and $\sin^2\theta+\cos^2\theta=1$. | $= m\sqrt{\left(\frac{v^2}{l}\right)^2 - 2\left(\frac{v^2}{l}\right)\left(g\sin\theta\right) + \left(g\right)^2(1)}$ |
| Factor $g^2$ out of the whole expression. | $= \boxed{mg\sqrt{\left(\frac{v^2}{gl}\right)^2 - 2\left(\frac{v^2}{gl}\right)\left(\sin\theta\right) + 1}}$ |

Interestingly, for a given angle, the answer only depends on two quantities: the weight of the cab and the ratio $v^2/gl$ .

# Formula Sheet

## Definitions

Position $x$

Velocity $v = \frac{dx}{dt}$

Acceleration $a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$

## Constant Acceleration

$$\vec{v} = \vec{v}_0 + \vec{a}t$$

$$\vec{x} = \vec{x}_0 + \vec{v}_0 t + \tfrac{1}{2}\vec{a}t^2$$

$$v^2 = v_0^{\,2} + 2a(x - x_0)$$

## Relative Motion

$$\vec{v}_{A,B} = \vec{v}_{A,E} + \vec{v}_{E,B}$$

$$\vec{v}_{E,B} = -\vec{v}_{B,E}$$

## Newton's Laws

$$\vec{F}_{\text{net}} = \sum_i \vec{F}_i = m\vec{a}$$

## Frictional Forces

$$f_{\text{k}} = \mu_{\text{k}} N$$

$$f_s \le \mu_s N$$

## Constants and Conversions

$$g = 9.81\ \tfrac{\text{m}}{\text{s}^2} = 32\ \tfrac{\text{ft}}{\text{s}^2}$$

$$1\ \text{mile} = 1.609\ \text{km}$$

## Quadratic Formula

If $ax^2 + bx + c = 0$ then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$